MA 412 tuts

Tutorial-1 :

1. $|z_1 + z_2 + \cdots + z_n| \le |z_1| + \cdots + |z_n|$

equality above is called triangle inequality

now if $|z_1 + \cdots + z_n| = |z_1| + \cdots + |z_n|$

$\Rightarrow |z_1 + \cdots + z_n|^2 = (|z_1| + \cdots |z_n|)^2$

$\Rightarrow (z_1 + \cdots + z_n)\overline{(z_1 + z_2 + \cdots + z_n)} = \sum |z_i|^2 + \sum_{i>j} 2|z_i||z_j|$

$\Rightarrow \cancel{\sum |z_i|^2} + 2\sum_{i>j} Re(z_i \bar{z}_j) = \cancel{\sum |z_i|^2} + \sum_{i>j} 2|z_i||z_j|$

$\Rightarrow \sum Re(z_i \bar{z}_j) = \sum |z_i||z_j|$

and as

$Re(z_i \bar{z}_j) \le |z_i||z_j|$

we if $Re(z_i \bar{z}_j) = |z_i||z_j|$

then

$\sum Re(z_i \bar{z}_j) = \sum |z_i||z_j|$

$\Rightarrow Re(z_i \bar{z}_j) = |z_i||z_j| \quad \forall i, j$ (as $i<j$ and $i>j$ both cases same)

$\Rightarrow Re\left(\frac{z_i}{z_j} \cdot z_j \bar{z}_j\right) = |z_i||z_j|$

$\Rightarrow |z_j|^{\cancel{2}} Re\left(\frac{z_i}{z_j}\right) = |z_i|\cancel{|z_j|}$

$\Rightarrow Re\left(\frac{z_i}{z_j}\right) = \frac{|z_i|}{|z_j|}$

or $\frac{z_i}{z_j}$ is Real and positive

2. $f: \mathbb{C} \to \mathbb{C}$ is cont at a or:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t

$|f(x) - f(a)| < \varepsilon$ for $\forall x$ s.t $|x - a| < \delta$

now, $|f| : \mathbb{C} \to \mathbb{R}_{\ge 0}$

$|f(x) - f(a)| \ge ||f(x)| - |f(a)||$ by triangle inequality

$\Rightarrow ||f(x)| - |f(a)|| < \varepsilon$

$\Rightarrow \lim_{x \to a} |f(x)| = |f(a)|$

$\Rightarrow |f|$ is cont at a.

3. $u : \mathbb{R}^2 \to \mathbb{R}$

say $u(x,y)$ is cont. differentiable

then $u(x+h, y+k) - u(x,y)$

$= u(x+h, y+k) - u(x+h, y) + u(x+h, y) - u(x,y)$

for $\sqrt{h^2+k^2} \to 0$

$|(h,k)| \to 0$

we

$|h| \le \sqrt{h^2+k^2}$

& $|k| \le \sqrt{h^2+k^2}$

now, $\displaystyle\lim_{\sqrt{h^2+k^2}\to 0} U(x+h, y+k) - U(x,y)$

$$= \lim_{\sqrt{h^2+k^2}\to 0} \left(\frac{U(x+h,y+k)-U(x+h,y)}{k}\right).k$$

$$+ \lim_{\sqrt{h^2+k^2}\to 0} \left(\frac{U(x+h,y)-U(x,y)}{h}\right).h$$

$$= \lim_{k\to 0}\left[\frac{U(x+h,y+k)-U(x+h,y)}{k}\right].k$$

$$+ \lim_{h\to 0}\left[\frac{U(x+h,y)-U(x,y)}{h}\right].h$$

$$= \lim_{\sqrt{h^2+k^2}\to 0}\left(\left[\frac{\partial}{\partial y} U(x+h,y)\right].k + \left[\frac{\partial}{\partial x} U(x,y)\right].h\right)$$

now $\frac{\partial}{\partial y} U(x+h,y)$ is cont

so for $h \to 0$

$$\lim_{h\to 0} \frac{\partial}{\partial y} U(x+h,y) = \frac{\partial}{\partial y} U(x,y)$$

so $\forall \varepsilon > 0, \exists \delta > 0$ s.t

$$\left|\frac{\partial}{\partial y} U(x+h,y) - \frac{\partial}{\partial y} U(x,y)\right| < \varepsilon$$

for $|(x+h,y)-(x,y)| < \delta$

$\Rightarrow |h| < \delta$

$$\Rightarrow \left|\frac{\partial}{\partial y} U(x+h,y)\right| < \varepsilon + \left|\frac{\partial}{\partial y} U(x,y)\right|$$

$$\Rightarrow \left|\frac{\partial}{\partial y} U(x+h,y)\right|.h < \varepsilon.h + \left|\frac{\partial}{\partial y} U(x,y)\right|.h$$

$\Rightarrow \exists\ O(x,y)$

Here LMVT can also be used when like Ball

$$\left|\frac{\partial}{\partial y} U(x+h,y)\right|.h = O(x,y) + \left|\frac{\partial}{\partial y} U(x,y)\right|.h$$

where $O(x,y) = \varepsilon'.h$ where $\varepsilon' < \varepsilon$ for all x, y

and $\underbrace{U(x+h,y)-U(x,y)}_{} = U'(x+\tilde{h},y)\tilde{h}$

in Ball

and now as $|h| < \delta$ and $|h| \to 0$ well $|h| < \delta$ always true

$$\therefore \left|\frac{\partial}{\partial y} U(x+h,y)\right| = O(x,y) + \left|\frac{\partial}{\partial y} U(x,y)\right|.h$$

and also $|h| \leqslant \sqrt{h^2+k^2} \Rightarrow \dfrac{|h|}{\sqrt{h^2+k^2}} \leq 1$

$$\Rightarrow \frac{O(x,y)}{\sqrt{h^2+k^2}} \to 0 \text{ as } \varepsilon \to 0$$
for $|h| \to 0$

let $O(x,y) = \gamma_{x,y}(h,k)$

4. for checking $f: \mathbb{R}^2 \to \mathbb{R}^2$ is real diff or not, let's find the T s.t. at a

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - T(h)|}{|h|} = 0$$

now $f(a+h) = (a_1 + h_1, -a_2 - h_2)$
$f(a) = (a_1, -a_2)$

$$\lim_{h\to 0} \frac{|(h_1, -h_2) - T(h)|}{|h|} = 0 \quad \text{for } T: \mathbb{R}^2 \to \mathbb{R}^2$$

if $T(h_1, h_2) = (h_1, -h_2)$
then we are done

so for $T(h) = \bar{h}$ we have
$Df = T$ or f is real diff

5. if $f(z) = \bar{z}$ is complex diff then

on both directions it is same
i.e it should satisfy C-R

$U_x = V_y$
$U_y = -V_x$

$f(x,y) = (x, -y)$

or $f(x+iy) = x - iy$
$= U(x,y) + iV(x,y)$

$U(x,y) = x$
$V(x,y) = -y$

$U_x = 1 \quad V_x = 0$
$U_y = 0 \quad V_y = -1$

$U_x \neq V_y \quad \therefore$ not true

for $z = a \in \mathbb{R}$

$f(a) = a \quad \therefore$ diff (trivial)

Tutorial-2:

1. $f(z) = \sum_{n\geq 1} n z^{n+10}$

and $z^{10}\left[\sum_{n\geq 1} n z^n\right]$ will have same radius of convergence

as $f(z)$ converges for $|z|<R$ then

$f(z) = z^{10} g(z)$ converges for $|z|<R$

$\Leftrightarrow$ $|z|\neq 0$ $\dfrac{f(z)}{z^{10}} = g(z)$ for $|z|<R$

now $g(z) = \sum n z^n$

$$R = \frac{1}{\limsup |a_n|^{1/n}} = \frac{1}{\limsup (n)^{1/n}}$$

now $\lim_{n\to\infty} (n)^{1/n} = 1$

as: let $\lim_{n\to\infty} (n)^{1/n} = 1+h$ then

$$(1+h)^n = n$$

$$\Rightarrow n = 1 + (n)h + \frac{(n)(n-1)}{2}h^2 + \cdots$$

$$> \frac{(n)(n-1)}{2}h^2$$

$$\Rightarrow \frac{2}{n-1} > h^2$$

$$\Rightarrow h^2 < \frac{2}{n-1}$$

as $n \to \infty$

$h^2 \to 0$

$\Rightarrow h \to 0$

$\therefore \lim_{n\to\infty} (n)^{1/n} = 1+h = 1$

$\Rightarrow \limsup (n)^{1/n} = \lim_{n\to\infty} (n)^{\frac{1}{n}} = 1$

$\Rightarrow R = 1 \Rightarrow \forall\, |z|<1$

$f(z)$ converges

2. $\{z_n\}_{n\geq 1}$ is Cauchy in complex numbers then $\lim_{n\to\infty} z_n = a+ib$

if $\{z_n\}_{n\geq 1}$ is Cauchy in complex numbers then

$\forall \varepsilon>0, \exists N \in \mathbb{N}$ s.t $\forall n,m \geq N$

$|z_n - z_m| < \varepsilon$

now $|z_n - z_m| < \varepsilon$

$\Rightarrow |Re(z_n) - Re(z_m) + i(Im(z_n) - Im(z_m))| < \varepsilon$

$\Rightarrow |Re(z_n) - Re(z_m)| < \varepsilon$ and $|Im(z_n) - Im(z_m)| < \varepsilon$

so when $\{z_n\}_{n\geq 1}$ is Cauchy
$\{Re(z_n)\}_{n\geq 1}$ and $\{Im(z_n)\}_{n\geq 1}$ is also Cauchy but in $\mathbb{R}$

so as $\mathbb{R}$ is complete
let

$$\lim_{n\to\infty} Re(z_n) = a$$

$$\lim_{n\to\infty} Im(z_n) = b$$

now, $\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ s.t
$\forall n \geq N_1$ $|Re(z_n) - a| < \varepsilon$

$\exists N_2 \in \mathbb{N}$ s.t
$\forall n \geq N_2$ $|Im(z_n) - b| < \varepsilon$

here $|Re(z_n) + i\,Im(z_n) - a - ib|$

$< |Re(z_n) - a| + |Im(z_n) - b|$

$< 2\varepsilon \quad \forall n \geq \max\{N_1, N_2\}$

$\therefore z_n \to a + ib$

3. $\sum_{n\geq 0} a_n z^n$ is a power series, $R > 0$

<u>To prove</u>: for $\rho < R$, the series converges uniformly in the domain $\{z \in \mathbb{C} \mid |z| \leq \rho\}$

<u>proof</u>: for $|z| \leq \rho < R$,
$\exists \rho'$ s.t
$\rho < \rho' < R$, now

$$|z| \leq \rho < \rho' < R = \frac{1}{\limsup |a_n|^{1/n}}$$

$$\frac{1}{R} = \limsup |a_n|^{1/n}$$

then $\frac{|z|}{R} \leq \frac{\rho}{R} < \frac{\rho'}{R} < 1 \Rightarrow \limsup |z|\,|a_n|^{1/n} < 1$

$\limsup |z||a_n|^{1/n} < 1$

$\exists N \in \mathbb{N}$ s.t $\forall n \geq N$

$|z||a_n|^{1/n} < 1$

$\Rightarrow |z|^n |a_n| < 1$
$\forall |z| \leq \rho$

$\frac{|z|}{R} < \frac{\rho'}{R} < 1$

$\Rightarrow \left(\frac{|z|}{R}\right)^n < \left(\frac{\rho'}{R}\right)^n < 1$

or $\exists N \in \mathbb{N}$ s.t $\forall n \geq N$

$|z||a_n|^{1/n} < \left(\frac{\rho'}{R}\right)^n < 1$

now, $\sum_{n\geq 0} a_n z^n = \underbrace{\sum_{n\geq 0}^{N} a_n z^n}_{\text{I}} + \underbrace{\sum_{n=N+1}^{\infty} a_n z^n}_{\text{II}}$

I : Finite so trivial

II : as for $n \geq N$, $\exists M_n = \left(\frac{\rho'}{R}\right)^n$
$\forall |z| \leq \rho$ and
$\sum M_n = c$ ← as infinite gp with $r = \frac{\rho'}{R} < 1$

and $|a_n z^n| < \left(\frac{\rho'}{R}\right)^n$

$\therefore \forall \sum u_n(z) \; \exists M_n$ s.t

$|u_n(z)| < M_n \;\forall |z| < \rho$ s.t

$\sum M_n$ is conv

$\therefore \sum u_n(z)$ is uniformly convergent

4. <u>To prove</u> : $f: \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-diff
$f'(z) = 0$ then f is constant

<u>proof</u>: as f is $\mathbb{C}$-diff, $\exists\, u(x,y), v(x,y)$ st

$f(x,y) = u(x,y) + i v(x,y)$

where

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\} \text{C-R equations}$$

as $f'(z)=0 \; \forall z \in \mathbb{C}$

for z on x-axis

$f'(z)=0$

$\underbrace{f(x,y)}_{x+iy=0} = u(x,0)+iv(x,0)$

$\frac{d}{\delta x} f(x,0) = u_x(x,0)+iv_x(x,0)=0$

$u_x(x,0)=0 \Rightarrow u$ is const in x axis

and

$v_x(x,0)=0 \Rightarrow v$ is const in x axis

C-R equations

$u_x = 0 = v_y \Rightarrow v$ is const in y axis

$u_y = -v_x \Rightarrow u_y=0 \Rightarrow u$ is const in y axis

this means $u(x,y)$ is const in x

$\Rightarrow u(x,y) = u'(y)$

and const in y

$\Rightarrow u(x,y) = c_1$

so $f = c_1 + ic_2$ → const, const

$\therefore f$ is const

5. $f(z)=z^2$

$f(z)=z^2$

$f(x,y) = (x+iy)^2 = x^2-y^2+2xyi$

$= u(x,y)+iv(x,y)$

$u(x,y) = x^2-y^2$

$v(x,y) = 2xy$

$u_x = 2x \quad u_y = -2y$

$v_x = 2y \quad v_y = 2x$

$\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\}$ C-R equations are verified

Tutorial-3·

1. $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$

$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$ } Defination of cos and sin

and we know

$$e^{z} = \sum_{i=0}^{\infty} \frac{(z)^i}{i!}$$

now also for $\theta = 2\pi$

$e^{i2\pi} = 1$
(proved in class)

then,

$$\cos(2\pi) = \frac{e^{i(2\pi)} + e^{-i(2\pi)}}{2}$$

$$= \frac{1+1}{2} = 1$$

$$\sin(2\pi) = \frac{1-1}{2i} = 0$$

now, $\cos(z) + i\sin(z) = \frac{e^{iz} + \cancel{e^{-iz}}}{2} + \frac{e^{iz} - \cancel{e^{-iz}}}{2}$

$= e^{iz}$

now
$e^{iz} = \cos(z) + i\sin(z)$
and

$$\cos^2(z) = \frac{e^{2iz}}{4} + \frac{e^{-2iz}}{4} + 2\frac{e^{iz}}{2}\frac{e^{-iz}}{2}$$

$$\sin^2(z) = -\frac{1}{4}\left(e^{2iz} + e^{2iz} - 2e^{iz}e^{-iz}\right)$$

$$\cos^2(z) + \sin^2(z) = \frac{1}{4}\left(\cancel{e^{2iz}} \cancel{+ e^{-2iz}} + 2 + 2\right)$$

$= 1$

so $\cos^2(z) + \sin^2(z) = 1$

now 2π we know

$e^{i2\pi} = 1$
$e^{i\pi} = \cos(\pi) + i\sin(\pi) = \pm 1$
as $(e^{i\pi})^2 = 1 = e^{i2\pi}$

$e^{i\pi} = \pm 1$

$(e^{i\pi/2})^2 = \pm 1 = e^{i\pi}$

$e^{i\pi/2} = 1$ or i

$e^{i\pi/2} \neq 1$ (as 2π is smallest)

$\Rightarrow e^{i\pi/2} = i = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$

$\cos\left(\frac{\pi}{2}\right) = 0$

$\sin\left(\frac{\pi}{2}\right) = 1$

now as $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$$e^{2i\theta} = \cos(2\theta) + i\sin(2\theta) = \cos^2\theta - \sin^2\theta + 2i\sin(\theta)\cos(\theta)$$

$\cos^2\left(\frac{\pi}{4}\right) - \sin^2\left(\frac{\pi}{4}\right) = 0$

$\Rightarrow \cos^2\left(\frac{\pi}{4}\right) = \sin^2\left(\frac{\pi}{4}\right)$

and $2\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) = 1$

$\Rightarrow \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) = \frac{1}{2}$

$\Rightarrow \cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \pm\frac{1}{\sqrt{2}}$

now as $\sin(x)$ is inc for $x > 0$ and for $\sin\left(\frac{\pi}{2}\right) = 1$

$\sin\left(\frac{\pi}{4}\right) < \sin\left(\frac{\pi}{2}\right)$

$\Rightarrow \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

and so $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

2. $i^2 = -1$

$$(-1)^i = \exp(i\,\mathrm{Log}(-1))$$
$$= \{\exp(i(\log|(-1)| + i(\pi) + 2\pi n i))\}_{n\in\mathbb{Z}}$$
$$= \{\exp(i)(\pi i + 2n\pi i)\}_{n\in\mathbb{Z}}$$

$= \{ \exp(-\pi - 2n\pi) \}_{n \in \mathbb{Z}}$

$i^{2i} = \exp(2i \log(i))$

$= \{ \exp(2i)(\frac{\pi}{2} i + 2\pi n i) \}_{n \in \mathbb{Z}}$

$= \{ \exp(-\pi - 4\pi n) \}_{n \in \mathbb{Z}}$

set of values are <u>different</u>

3. $|\Sigma z_j w_j|^2 \leqslant \Sigma |z_j|^2 \Sigma |w_j|^2$

<u>To prove</u>: $|ab + bc + ac| \leqslant |a|^2 + |b|^2 + |c|^2$

<u>proof</u>: $z_1 = a \quad b = \omega_1$
$z_2 = b \quad c = \omega_2$
$z_3 = c \quad a = \omega_3$

$|\Sigma z_i w_i|^2 \leqslant \Sigma |z_i|^2 \Sigma |w_i|^2$

$|ab + bc + ca|^2 \leqslant (|a|^2 + |b|^2 + |c|^2)(|a|^2 + |b|^2 + |c|^2)$

$\Rightarrow |ab + bc + ca| \leqslant |a|^2 + |b|^2 + |c|^2$

4. $f(z) = |z|$

if $f(z)$ is diff at 0 then:

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(h) - f(0)}{h}$$

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(h) - f(0)}{h} = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{|h|}{h}$$ does not exist

so <u>not</u> diff at $z = 0$

now for $z \neq 0$

$f(z) = |z| \Rightarrow \sqrt{x^2 + y^2} = f(x, y)$

$u(x, y) = \sqrt{x^2 + y^2}$
$v(x, y) = 0$

then u, v are cont when $(x, y) \neq 0$
and partial derivatives of v is also is cont and exist for $(x, y) \neq 0$ (trivial)

(cont is not needed)

$u_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}}$ $\quad v_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}}$ then this is also cont for $(x, y) \neq (0, 0)$

now, if $u_x = v_y$, $u_y = -v_x$ $\left((x,y) \neq (0,0)\right)$

then f is C-diff

here

$u_x = v_y \Rightarrow \dfrac{x}{\sqrt{x^2+y^2}} = 0$

(f is C-Diff $\Rightarrow$ C-R eq satisfied)

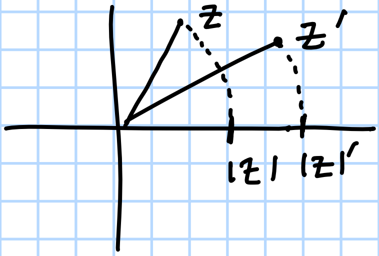
$\Rightarrow x = 0$

and $u_y = -v_x \Rightarrow y = 0$

this is not possible for $(x,y) \neq (0,0)$

so no such point exist

$f(z) = |z|$



($\sim$ C-R eq satisfied $\Rightarrow \sim f$ is C-Diff)

5. $f(z) = \dfrac{1}{(1-z)^2}$ centred at $z = 0$

$f(z) = \dfrac{1}{(1-z)^2}$ cented at $z = 0$

now for $g(z) = 1 + z + z^2 + \cdots\cdots$

$\overline{\lim} |1|^{\frac{1}{n}} = \overline{\lim}(1) = 1$

$R = \dfrac{1}{\overline{\lim}(1)} = 1$

or $|z| < 1 \Rightarrow g(z) = 1 + z + z^2 + \cdots$ is convergent

now

$zg(z) = z + z^2 + \cdots\cdots$
$= g(z) - 1$

$1 = (1-z)\,g(z)$

$\Rightarrow g(z) = \dfrac{1}{1-z}$ for $|z| < 1$

now, diff of $g(z)$ will give us

$g'(z) = \dfrac{1}{(1-z)^2} = \dfrac{\partial}{\partial z}\left(1 + z + \cdots\cdots\right)$
$= 1 + 2z + 3z^2 + \cdots\cdots$

as $g(z)$ has $R = 1 \Rightarrow g'(z)$ also has $R = 1$

and for $|z|<1$ we have

$$\frac{1}{(1-z)^2} = 1+2z+3z^2+\ldots\ldots$$

$$\Rightarrow \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty}(n)z^{n-1}$$

power series of $\frac{1}{(1-z)^2}$

centered at 0

Radius of conv $=1$

Assignment - 1:

23B3321
Dhairya Kantawala
dhairya@iitb.ac.in

1. To show : C-R equations are verified in case of e^z

now, $e^z = \exp(z)$
where $z = x+iy$
$x, y \in \mathbb{R}$
$\Rightarrow$
$e^z = e^{x+iy}$
$= e^x(e^{iy})$

now by defination $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$

$\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$

and as $e^z = \sum_{n=0}^{\infty} \frac{(z)^n}{n!}$

$$\frac{d}{dz} e^z = \sum_{n=1}^{\infty} \frac{n(z)^n}{n!} = \sum_{n'=0}^{\infty} \frac{(z)^{n'}}{(n')!} = e^z$$

so $\frac{d}{dz} e^z = e^z$

$$\therefore \frac{\partial \cos(z)}{\partial z} = \frac{e^{iz}(i) - e^{-iz}(-i)}{2}$$

$= -\sin(z)$

and sim

$\frac{\partial}{\partial z} \sin(z) = \cos(z)$

now $e^z = e^x(e^{iy})$
and
now $\cos(y) + i\sin(y)$

$$= \left(\frac{e^{iy}+e^{-iy}}{2}\right) + i\left(\frac{e^{iy}-e^{-iy}}{2}\right)$$

$= e^{iy}$

now $e^z = e^x(\cos y + i\sin y)$

$= e^x\cos y + ie^x\sin y$

where
$x, y \in \mathbb{R}$

$\exp(x+iy) = e^x\cos y + ie^x\sin y$

$u(x,y) = e^x\cos y$
$v(x,y) = e^x\sin y$

now $u_x = \frac{\partial}{\partial x}(e^x\cos y)$ (from $\frac{\partial}{\partial x} e^x = e^x$)

$u_x = e^x\cos y$ — ①
$u_y = -e^x\sin y$ — ② (from $\frac{\partial}{\partial y}\cos y = -\sin y$)

$V_x = e^x \sin(y)$ — ③
$V_y = e^x \cos(y)$ — ④
} similar to ①, ②

then $U_x = V_y$
as ① = ④
and
$U_y = -V_x$ as ② = −③

∴ C-R equations are verified

2. $z \in \mathbb{C}^\times$ given, $n \in \mathbb{N}$

<u>To prove</u>: There are exactly n distinct complex numbers satisfying $w_j^n = z$ for $j = 1, 2 \ldots n$

<u>proof</u>: using (1) we can write

$$z = e^{x+iy} = e^x(e^{iy})$$

$y = \theta$ & $e^x = r$
then

$$z = r(e^{i\theta})$$

for some $r = e^x > 0$
as $x, y \in \mathbb{R}$
e^x is always positive
(proved in class)

$r > 0$
and also $y \in \mathbb{R} \Rightarrow \theta \in \mathbb{R}$

now $z = re^{i\theta}$
for some $r > 0$, $\theta \in \mathbb{R}$

also we have proved in class that $\exists\, \theta' = 2\pi$
s.t

$e^{i2\pi} = 1$

i.e $e^{i\theta}$ is periodic

then $z = re^{i\theta + 2\pi i k}$
for $k \in \mathbb{Z}$

now we fix θ' s.t

$\theta' \in [0, 2\pi)$

s.t $\theta' + 2\pi k' = \theta + 2\pi k$

this is possible as

if $\theta \in [2\pi n', 2\pi n' + 2\pi)$
then

$(\theta - 2\pi n') \in [0, 2\pi)$

let $\theta - 2\pi n' = \theta'$

so $z = r e^{i\theta' + i2\pi k'}$
for $\theta' \in [0, 2\pi)$
$k' \in \mathbb{Z}$

now $w^n = z = r e^{i\theta' + i2\pi k'}$

$w = (r)^{\frac{1}{n}} e^{\frac{i\theta'}{n} + \frac{i2\pi k'}{n}}$

then $\frac{\theta'}{n} \in [0, \frac{2\pi}{n})$

and $\frac{2\pi k'}{n}$

now as $\frac{2\pi k'}{n}$ is for all $k' \in \mathbb{Z}$

for $k' = 1 \Rightarrow \frac{2\pi}{n}$

$k' = 2 \Rightarrow \frac{2\pi}{n}(2)$

$\vdots$

$k' = n \Rightarrow \frac{2\pi}{\cancel{n}}(\cancel{n}) = 2\pi$

or we see that if $w = (z)^{\frac{1}{n}} = \underbrace{(r)^{\frac{1}{n}}}_{\substack{>0 \\ \in \mathbb{R}}} \times (e^{\frac{i\theta'}{n} + i2\frac{\pi k'}{n}})$ for $k' \in \mathbb{Z}$

then

$\left\{ \frac{\theta'}{n} + \frac{2\pi k'}{n} \right\}_{k' \in \mathbb{Z}}$

has n cases

$$\left.\begin{aligned} \text{for } k' &= 1 + n\overset{\in \mathbb{Z}}{k''} \\ k' &= 2 + nk'' \\ &\vdots \\ k' &= \overset{n-1}{\cancel{0}} + nk'' \end{aligned}\right\} n \text{ cases}$$

or $\left\{ \frac{\theta'}{n} + \frac{2\pi k'}{n} \right\}_{k' \in \mathbb{Z}} = \left\{ \frac{\theta'}{n} + 2\pi k'', \frac{\theta'}{n} + \frac{2\pi}{n} + 2\pi k'', \right.$

$\left. \ldots\ldots, \frac{\theta'}{n} + \frac{2\pi}{n}(n-1) + 2\pi k'' \right\}_{k'' \in \mathbb{Z}}$

and $z^{\frac{1}{n}} = (r)^{\frac{1}{n}} e^{i\theta/n}$

from above will have the following cases:

$\left\{ (r)^{\frac{1}{n}} e^{i\theta'/n + \frac{i2\pi k''}{n}}, \ldots\ldots (r)^{\frac{1}{n}} e^{i\theta'/n + i2\pi/n(n-1) + i2\pi k''} \right\}_{k'' \in \mathbb{Z}}$

but as $e^{i2\pi k''} = 1$

$\Rightarrow \left\{ r^{\frac{1}{n}} (e^{i\theta'/n + i2\pi/n (k''')}) \right\}$ for $k''' = 0, 1, \ldots n-1$

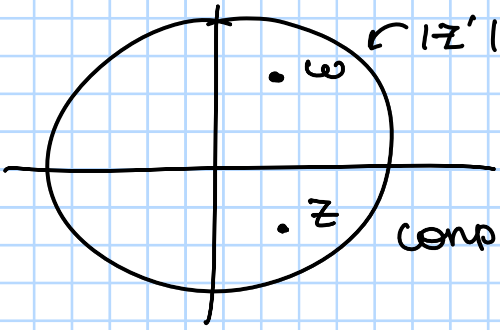
$\therefore$ ω will have n values of :

$$\omega_j = (r)^{1/n} e^{i\theta/n} e^{i2\pi/n \times (j-1)}$$
for $j = 1, 2 \ldots n$

3. given $z, w \in \mathbb{C}$, $z\bar{w} \neq 1$

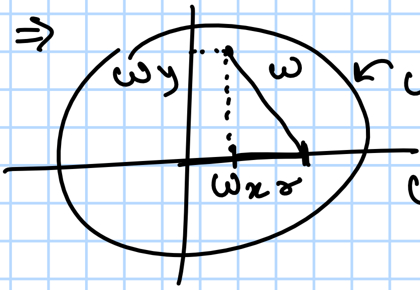
To prove : $\left|\dfrac{z-w}{1-z\bar{w}}\right| < 1$ for $|z|, |w| < 1$

proof :



now for $z = r \geq 0 \in \mathbb{R}$

$\Rightarrow$



now, $|r - \omega| =$ dist b/w ω and r

$$\omega = \omega_x + i\omega_y$$

$$|r-\omega| = \sqrt{\omega_y^2 + (r-\omega_x)^2}$$

now $|r-\omega|^2 = (r-\omega)(r-\bar{\omega})$

$$= \omega_y^2 + (r-\omega_x)^2 = \omega_y^2 + \omega_x^2 + r^2 - 2r\omega_x$$

now, $|1 - r\omega| =$ dist b/w $(1,0)$ and $r\omega$
ω scaled by r

$$|1 - r\omega| = \sqrt{(1-r\omega_x)^2 + (r\omega_y)^2}$$

now as $z = r$
$|r| < 1$
$|\omega| < 1$

we have $|r|^2 = a$
$|\omega|^2 = b$

then
$a < 1$, $b < 1 \Rightarrow (1-a) > 0$, $(1-b) > 0$

$$\Rightarrow (1-a)(1-b) > 0$$

$\Rightarrow (1-a-b+ab) > 0$

$\Rightarrow 1+ab > a+b$

now $a = |r|^2 = r^2$
$b = |w|^2 = w\bar{w}$

$\Rightarrow 1+r^2 w\bar{w} > r^2 + w\bar{w}$
$\Rightarrow 1+r^2 w\bar{w} - wr - r\bar{w} > r^2 + w\bar{w} - wr - r\bar{w}$

by adding $-wr - r\bar{w}$ on both side

$\Rightarrow (1-rw)(1-r\bar{w}) > (r-w)(r-\bar{w})$

so we get $|1-rw|^2 > |r-w|^2$

$\Rightarrow |1-rw| > |r-w|$

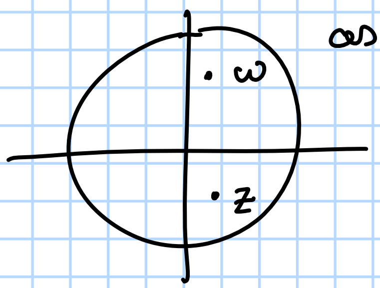
so, the dist b/w $(r,0)$ and (w_x, w_y) is less than $(1,0)$ and (rw_x, rw_y)

when $\sqrt{r^2+0^2} < 1$
$\sqrt{w_x^2 + w_y^2} < 1$

now, so $\left|\frac{r-w}{1-r\bar{w}}\right| < 1$ for $z = r \geq 0$, $r \in \mathbb{R}$

(as $r\bar{w} \neq 1$) (given)

now let's generlise this for $z \in \mathbb{C}$



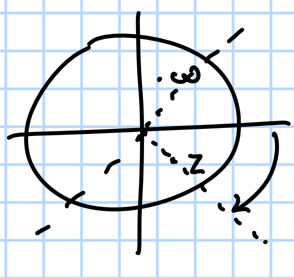
as $|w| < 1$
$|z| < 1$

and $|w - z|$ is dist b/w w and z

$|1 - z\bar{w}|$ is dist b/w $(1,0)$ and $(z\bar{w})$

we can rotate the plane and still the distance will remain same

$\Rightarrow$



we rotate s.t z is on new x-axis

so z in new plane ≥ 0 and $\in \mathbb{R}$

so, $\left|\frac{z-w}{1-z\bar{w}}\right| < 1$ in new rotated frame, as distance remains same even after rotation

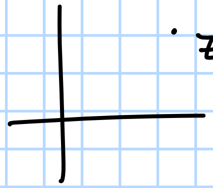
$\left|\frac{z-\omega}{1-z\bar{\omega}}\right|<1$ in old plane

$\forall z \in \mathbb{C}$ s.t

$|z|<1$, $z\bar{\omega} \neq 1$

now proof of rotation preserves distance

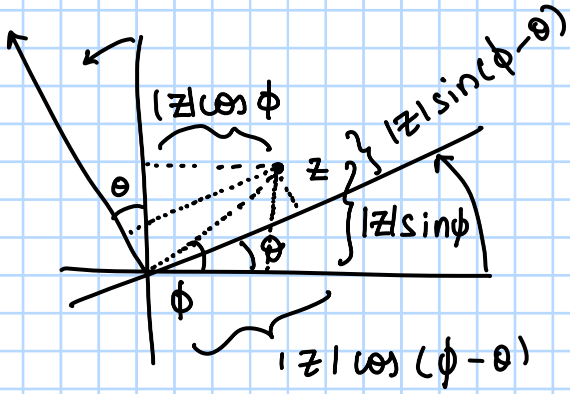
the



$z = x+iy$ can be written as (x,y) and

$|z| = \sqrt{x^2+y^2}$

now by rotating the plane anticlockwise by θ:



new coordinates: $(|z|\cos(\phi-\theta), |z|\sin(\phi-\theta))$

old coordinates: $(|z|\cos(\phi), |z|\sin(\phi))$

now $|\text{new coordinates}| = |z|\sqrt{\cos^2(\phi-\theta)+\sin^2(\phi-\theta)}$

$= |z|$

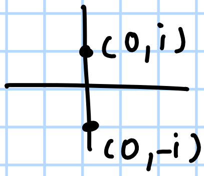
$|\text{old coordinates}| = |z|\sqrt{\cos^2(\phi)+\sin^2(\phi)}$

$= |z|$

so old coord dist = new coord dist

Tutorial – 4:

1. $\int_{\gamma} \bar{z}\, dz$ γ is st line from $-i$ to i



$\gamma(t) = i(2t-1)$ → Path from $[0,1] \to [-i,i]$

now $\int_{\gamma(t)} \bar{z}\, dz$

$\bar{z} = f(z)$

$f(\gamma(t)) = \overline{i(2t-1)}$

$= i(1-2t)$

$\int_{\gamma(t)} f(\gamma(t))\, \gamma'(t)\, dt$

$= \int_0^1 i(1-2t)(i)(2)\, dt$

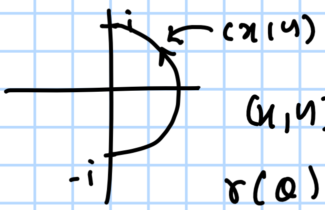
$= \int_0^1 (2t-1)(2)\, dt$

$= \int_0^1 4t-2\, dt$

$= \left| 4\frac{t^2}{2} - 2t \right|_0^1 = 2\left[(t^2-t)\right]_0^1$

$= 2[(1-1)-(0)]$
$= 0$

2. $\int_{\gamma} \bar{z}\, dz$



$(x,y) = (\cos\theta, \sin\theta)$

$\gamma(\theta) = \cos\theta + i\sin\theta$

$\theta \in (-\pi/2, \pi/2)$

$\int_{-\pi/2}^{\pi/2} (\cos\theta - i\sin\theta)(-\sin\theta + i\cos\theta)(d\theta)$

$= \int_{-\pi/2}^{\pi/2} (\cancel{-\cos\theta\sin\theta} + i + \cancel{\sin\theta\cos\theta})\, d\theta$

$= i\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = i\pi$

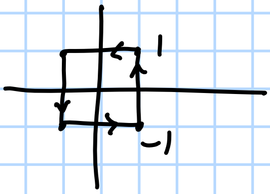
3. $\int_{\gamma} \frac{1}{z}\, dz$ $z=1$ to $z=i$

$\theta \in (0, \pi/2)$

$\int_0^{\pi/2} \cancel{e^{-i\theta}}(\cancel{e^{i\theta}})(i)(d\theta)$

$= \int_0^{\pi/2} i\, d\theta = \frac{i\pi}{2}$

4. $\int_\gamma \frac{1}{z}\, dz$ in retagle



$\int_\gamma \frac{1}{z}\, dz$ as

for $\gamma(t) = (1, 2t-1) = 1 + i(t)$

$$\int_1^{-1} \frac{1}{(t+i)}(1)\,dt + \int_{-1}^{1} \frac{1}{(1+it)}(i)(dt)$$

$$+\int_1^{-1} \frac{1}{(-1+it)}(i)\,dt + \int_{-1}^{1}\frac{1}{t-i}(1)\,dt$$

$$= \int_{-1}^{1} \left(\frac{1}{t-i}\right) - \left(\frac{1}{t+i}\right) dt + \int_{-1}^{1}\left(\frac{i}{1+it}\right)+\left(\frac{i}{1-it}\right) dt$$

$$= \int_{-1}^{1} \left(\frac{\cancel{t}+i-\cancel{t}+i}{t^2+1}\right) dt + \int_{-1}^{1} i\left(\frac{1-\cancel{it}+1+\cancel{it}}{1+t^2}\right) dt$$

$$= \int_{-1}^{1} \frac{2i}{t^2+1}\,dt + \int_{-1}^{1} \frac{2i}{1+t^2}\,dt$$

$$= 4i\int_{-1}^{1} \frac{1}{t^2+1}\,dt$$

$$= 4i\left|\tan^{-1} t\right|_{-1}^{1}$$

$$= 4i[\tan^{-1}(1) - \tan^{-1}(-1)]$$

$$= 4i\left[\frac{\pi}{4}+\frac{\pi}{4}\right]$$

$$= 2\pi i$$

4.5 $\int \frac{1}{z^2}\, dz$ $\quad z = \cos\theta + i\sin\theta$

$$\int_0^{2\pi} e^{-i2\theta}(e^{i\theta})(i)\,d\theta$$

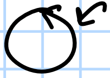
$$= \int_0^{2\pi} e^{-i\theta}(i)\,d\theta$$

$$= i\int_0^{2\pi}(\cos\theta - i\sin\theta)\,d\theta$$

$$= i\left[\sin\theta + i\cos\theta\right]_0^{2\pi}$$

$$= i[0-0+i(\cancel{1}-\cancel{1})]$$

$$= 0$$

also as circle  closed

Tutorial-5 :

5.1 $u(x,y) = \sinh(x)\cos(y)$

① u is harmonic
as
$$u_x = \cosh(x)\cos(y)$$
$$u_{xx} = \sinh(x)\cos(y)$$
$$u_y = -\sinh(x)\sin(y)$$
$$u_{yy} = -\sinh(x)\cos(y)$$

$$u_{xx} + u_{yy} = 0$$

② now let $v(x,y) = \int_b^y u_x(x,t)\,dt$

$$= \int_b^y \cosh(x)\cos(t)\,dt$$

$$v(x,y) = \int_b^y \cosh(x)\cos(t)\,dt$$

$$v(x,y) = \cosh(x)\sin(y) + \phi(x)$$

now
$$v_y = \cosh(x)\cos(y)$$
$$= u_x \quad \text{—①}$$

now, $v_x = \sinh(x)\sin(y) + \phi'(x)$
$= -u_y$
then $\phi'(x) = 0$
$\Rightarrow \phi(x) = c$

$\therefore v(x,y) = \cosh(x)\sin(y) + c$

now
$$v_{xx} = \cosh(x)\sin(y)$$
$$v_{yy} = -\cosh(x)\sin(y)$$

$$u_x = v_y$$
$$u_y = -v_x$$

$\therefore v$ is harmonic conjugate

5.2 $u(x,y) = x^3 - xy^2$

$$u_x = 3x^2 - y^2$$
$$u_{xx} = 6x$$

$$u_y = -2xy$$
$$u_{yy} = -2x$$
$$u_{xx} + u_{yy} = 4x \neq 0$$
so. $\nabla . u \neq 0$
$\therefore u$ is not harmonic

$\Rightarrow u$ cannot be real part of holomorphic funtion
as $u_{xx} = (u_x)_x$
$= (v_y)_x$
$u_{yy} = (-v_x)_y = -v_{yx} = -u_{xx} \Rightarrow u_{xx} + u_{yy} = 0$

5.3 $u(x,y) = ax^2 + bxy + cy^2$

again if u is harmonic

i.e $u \in C^2(\mathbb{R}^2)$

and $u_{xx} + u_{yy} = 0$

$\Rightarrow u_{xx} = 2a$

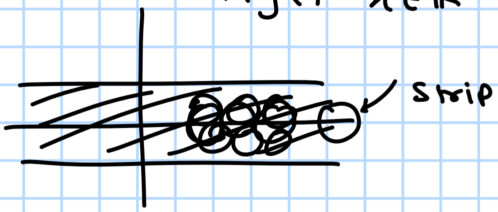
$u_{yy} = 2c$

$u_{xx} + u_{yy} = 2(a+c) = 0$

$\Rightarrow a + c = 0$

5.4 $f \leftarrow$ holomorphic

$-1 < y < 1 \quad x \in \mathbb{R}$



$|f(z)| \leq A(1+|z|)^b$

$\forall z$ in the strip

now $f^{(n)}(z) = \dfrac{n!}{2\pi i} \displaystyle\int_{\delta D} \frac{f(w)}{(w-z)^{n+1}} dw$

for $z \in \text{int}(D)$

and $D \subseteq$ strip $\rightarrow$ closed

now, $|f^{(n)}(z)| = \left| \dfrac{n!}{2\pi i} \displaystyle\int_{\delta D} \frac{f(w)\, dw}{(w-z)^{n+1}} \right|$

$\leq \dfrac{n!}{(R)^n} \left| \dfrac{1}{2\pi i} \displaystyle\int_{\delta D} \frac{f(w)}{(w-z)} dw \right|$

now $\dfrac{1}{2\pi i} \displaystyle\int_{\delta D} \frac{f(w)}{(w-z)} dw = f(z)$

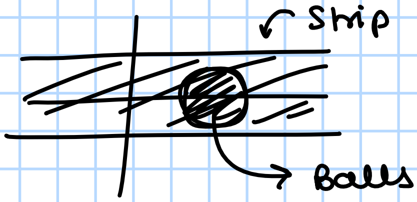
$\Rightarrow |f^{(n)}(z)| \leq \dfrac{n! \, |f(z)|}{R^n} \leq \dfrac{n!}{R^n} A(1+|z|)^b$

$\Rightarrow |f^n(x)| \leq \dfrac{n!}{R^n} A(1+R+|x|)^b$

$\Rightarrow |f^n(x)| \leq \dfrac{n!}{R^n} A(1+R+|x|+R|x|)$

$\Rightarrow |f^n(x)| \leq \dfrac{n!}{R^n} A(1+R)^b(1+|x|)^b$

$A_n = \dfrac{n!}{R^n}(1+R)^b \quad R = \dfrac{1}{2}$ works



5.5 $f(z) = \bar{z}$ on a closed disk $|z| \le 1$

then

① f is cont (trivial)

② $f(z) = \bar{z}$

then if $f(z)$ can be approximated as polynomial then

$$|\bar{z} - p_\varepsilon(z)| < \varepsilon$$

$p_\varepsilon \to$ the polynomial

now $\frac{1}{2\pi i} \int_{|z|=1} \bar{z} \, dz = \frac{1}{2\pi i} \int_0^{2\pi} \cancel{e^{-it}} \cancel{e^{it}} \, dt \, i = 1$

now $\varepsilon > 0$, $1 = \left| \frac{1}{2\pi i} \int_{|z|=1} \bar{z} \, dz \right|$

$$= \left| \frac{1}{2\pi i} \int_{|z|=1} \bar{z} - p_\varepsilon(z) \, dz + \underbrace{\frac{1}{2\pi i} \int_{|z|=1} p_\varepsilon(z) \, dz}_{0} \right|$$

$$\le \frac{1}{\cancel{2\pi}} \times \cancel{2\pi} \times \sup_{|z|=1} |p_\varepsilon(z) - \bar{z}| < \varepsilon$$

for $\varepsilon < 1$ this is a contradiction

so not possible for $f(z) = \bar{z}$

Tutorial-6 :

6.1 $f(z) = \dfrac{1}{i+z}$ around $z=-i$

now lets see on what Ω is f holomorphic
as

$$f(z) = \frac{1}{i+z}$$

$$f(x,y) = \frac{1}{i+x+iy}$$

$$= \frac{x - i(y+1)}{x^2+(y+1)^2}$$

$$= u(x,y) + i v(x,y)$$

$$u(x,y) = \frac{x}{x^2+(y+1)^2}$$

$$v(x,y) = \frac{-(y+1)}{x^2+(y+1)^2}$$

now $u_x = \dfrac{1}{x^2+(y+1)^2} + \dfrac{(x)(-1)}{((x^2)+(y+1)^2)^2}(2x)$

$$= \frac{x^2+(y+1)^2-2x^2}{(x^2+(y+1)^2)^2}$$

$$= \frac{(y+1)^2-x^2}{(x^2+(y+1)^2)^2} \qquad v_y = \frac{(y+1)^2-x^2}{(x^2+(y+1)^2)^2}$$

$$u_y = \frac{x(-1)}{(x^2+(y+1)^2)^2} \times (2(y+1))$$

$$u_y = \frac{-2x(y+1)}{(x^2+(y+1)^2)^2} \qquad v_x = \frac{2x(y+1)}{(x^2+(y+1)^2)^2}$$

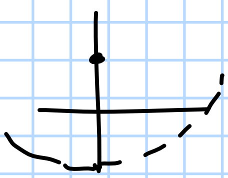
so, $u_x = v_y$ $\quad u_y = -v_x$ and u_x, v_y, u_y, v_x are cont
when $x^2+(y+1)^2 \neq 0$

$\Rightarrow x \neq 0, y \neq -1$

$\therefore$ f is holomorphic
except $z=-i$

now let $\Omega = \mathbb{C} \setminus \{-i\}$

let $D' = \{ (x)^2 + (y-1)^2 < 4 \}$



then $D' \subseteq \Omega$
and now

$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$$

case z_0 = centre of D'
= i

$$f(z) = \sum_{n\geq 0} \frac{f^{(n)}(i)(z-i)^n}{n!}$$

now as $f(z) = \frac{1}{i+z}$ $f(i) = \frac{1}{2i}$

$f^{(1)}(z) = \frac{-1}{(i+z)^2}$ $f^{(1)}(i) = \frac{-1}{(2i)^2}$

$f^{(2)}(z) = \frac{2}{(i+z)^3}$ $f^{(2)}(i) = \frac{2}{(2i)^3}$

$\vdots$ $\vdots$

$f^{(n)}(z) = \frac{(n)!\,(-1)^n}{(i+z)^{n+1}}$ $f^{(n)}(i) = \frac{(n)!\,(-1)^n}{(2i)^{n+1}}$

$$f(z) = \left(\frac{1}{2i}\right) \times \frac{1}{0!} \times (z-i)^0 + \sum_{n\geq 0} \frac{\cancel{n!}(-1)^n}{(2i)^{n+1}} \times \frac{1}{\cancel{n!}} \times (z-i)^n$$

$$= \frac{1}{2i} + \sum_{n\geq 1} \frac{(-1)^n}{(2i)^{n+1}} (z-i)^n$$

→ doubt at $z = -i$ not holomorphic

$$= \sum_{n\geq 0} \frac{(-1)^n}{(2i)} (z-i)^n$$

with $R = 2$

But for $z = -i$ no power expansion as not holomorphic at $z = -i$

6.2 $a, b \in \mathbb{C}$
s.t $a^2 + b^2 = 0$

__To prove__: $f(x,y) = (ax+by)^m$ is harmonic funtion for each $m \geq 1$

__proof__: $f_x = m(ax+by)^{m-1}(a)$

$f_{xx} = (m)(m-1)(ax+by)^{m-2}(a^2)$

$f_y = (m)(ax+by)^{m-1}(b)$

$f_{yy} = (m)(ax+by)^{m-2}(b^2)(m-1)$

$f_{xx} + f_{yy} = m(ax+by)^{m-2}(a^2+b^2)(m-1)$
$= 0$

if $m = 1$ then $f_x = a$
$f_{xx} = 0$
$f_{yy} = 0$
$\Rightarrow f_{xx} + f_{yy} = 0$

$\Rightarrow f$ is harmonic

6.3 $f(z)$ is non-zero holomorphic function then its zeroes cannot have a limit point

as if 0 has a limit point i.e

$\{x_n\}$ s.t $f(x_n) = 0$ $\forall x_n$

$x_n \longrightarrow x$

then $f \equiv 0$ which is not possible as f is non-zero

$\therefore$ $f(z)$ if non-zero harmonic then it's zeroes cannot have a limit point

$x_n \longrightarrow x$ for x_n being zeroes of f

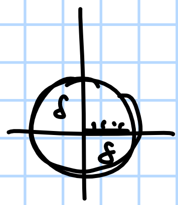
<u>To prove</u>: $f(z) = \sin\left(\frac{1}{z}\right)$ is non-holomorphic in any neighbourhood covering 0

<u>proof</u>: let f be holomorphic in $B_\delta(0)$ then let

$\Omega = B_\delta(0)$

this is a ball and hence connected

now,



now let $z = \frac{1}{n\pi}$

s.t $\frac{1}{n\pi} < \delta$

i.e $n > \frac{1}{\delta\pi}$

then let $n_1 = \left\lfloor \frac{1}{\delta\pi} \right\rfloor + 1$

$n_r = \left\lfloor \frac{1}{\delta\pi} \right\rfloor + r$

$n_i > \frac{1}{\delta\pi}$ $\forall i \in \mathbb{N}$

$\Rightarrow \frac{1}{z} = n_r\pi$ for r

now $\left\{\frac{1}{n_r\pi}\right\}_{r=1}^{\infty}$ is a seq s.t it converges to 0

and $f\left(\frac{1}{n_r\pi}\right) = \sin(n_r\pi) = 0$

now, $\left\{\frac{1}{n_r\pi}\right\}_{r=1}^{\infty} \longrightarrow 0$

and $f\left(\frac{1}{n_r\pi}\right) = 0 \Rightarrow f \equiv 0$ on ball $B_\delta(0)$

so for $B_\delta(0)$ $\quad \frac{1}{n_r\pi} \in B_\delta(0)$

and $\left\{ \frac{1}{n_r\pi} \right\} \longrightarrow 0 \in B_\delta(0)$

$f\left(\frac{1}{n_r\pi}\right) = 0 \quad \forall n \Rightarrow f \equiv 0$ on $B_\delta(0)$

now but as $f(z) = \sin\left(\frac{1}{z}\right)$ it is not $\equiv 0$ for $B_\delta(0)$

as for $z = \frac{1}{n_r\pi + \pi/2}$

$\sin\left(\frac{1}{z}\right) \neq 0$

now, this means that for any nbd of 0 making δ small enough will ensure $B_\delta(0) \subseteq$ Nbd

and as f is <u>not</u> holomorphic on $B_\delta(0)$

$\Rightarrow$ B is not holomorphic on Nbd

6.4 To find real and imaginary parts of $f(z) = \exp\left(\frac{1}{z}\right)$

$$f(x,y) = e^{\frac{1}{x+iy}}$$

$$f(x,y) = e^{\frac{x-iy}{x^2+y^2}}$$

$$f(x,y) = e^{\frac{x}{x^2+y^2}} e^{\frac{-iy}{x^2+y^2}}$$

$$= e^{\frac{x}{x^2+y^2}} \left[\cos\left(\frac{-y}{x^2+y^2}\right) + i\sin\left(\frac{-y}{x^2+y^2}\right)\right]$$

$$= e^{\frac{x}{x^2+y^2}} \left[\cos\left(\frac{y}{x^2+y^2}\right) - i\sin\left(\frac{y}{x^2+y^2}\right)\right]$$

$$= u(x,y) + iv(x,y)$$

$$\operatorname{Re} f(x,y) = u(x,y)$$

$$= e^{\frac{x}{x^2+y^2}} \cos\left(\frac{y}{x^2+y^2}\right)$$

$$\operatorname{Im} f(x,y) = e^{\frac{x}{x^2+y^2}} \sin\left(\frac{-y}{x^2+y^2}\right) = v(x,y)$$

6.5 <u>To prove</u>: $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ converges uniformly on $f(x) = |x|$ for $x \in [-1,1]$

<u>proof</u>: $f_n(x) = \sqrt{x^2 + \frac{1}{n}} \leq \sqrt{x^2 + \frac{2|x|}{\sqrt{n}} + \frac{1}{n}} \leq |x| + \frac{1}{\sqrt{n}}$ for $x \in [-1,1]$

now $|f_n(x) - |x||$
$= \left|\frac{1}{\sqrt{n}}\right|$

then $\forall \varepsilon > 0 \quad \left|\frac{1}{\sqrt{n}}\right| < \varepsilon$

let $n > \frac{1}{\varepsilon^2}$

for $N = \left\lfloor \frac{1}{\varepsilon^2} \right\rfloor + 1$

$\exists N \in \mathbb{N}$ s.t $\forall n \geq N$

$\Rightarrow \left| f_n(x) - f(x) \right| \leq \frac{1}{\sqrt{n}} < \varepsilon$

$\forall x \in [-1, 1]$

$\Rightarrow f_n(x) \xrightarrow{\text{unif}} f(x)$

Assignment - 2 :

Dhairya Kantawala
(23B3321)

1. $u(x,y) = x^3 - 3xy^2 + x$ is a function s.t

$$u : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

now $\frac{\partial}{\partial x} u(x,y) = \frac{\partial}{\partial x}(x^3) - \frac{\partial}{\partial x}(3xy^2) + \frac{\partial}{\partial x}(x)$

$$u_x = 3x^2 - 3y^2 + 1 \quad (\because \text{trivial differentiation})$$

now similary $\frac{\partial}{\partial y} u(x,y) = u_y = \frac{\partial}{\partial y}(-3xy^2)$ ($\because$ everything else const in y)

$$u_y = -6xy$$

now, $u_{xx} = \frac{\partial}{\partial x}(3x^2 - 3y^2 + 1) = 6x$

$u_{yy} = \frac{\partial}{\partial y}(-6xy) = -6x$

now from the continuity of u_{xx}, u_{yy} and $u_{xx} + u_{yy} = 0$ (u_{xx}, u_{yy} are cont as they are polynomials)

$\Rightarrow$ $u \in C^2(\mathbb{R}^2)$

and $\Delta . u = 0$ or u is harmonic function.

now, to find the harmonic conjugate,

i.e $V : \mathbb{R}^2 \longrightarrow \mathbb{R}$

s.t $V \in C^2(\mathbb{R}^2)$

$\Delta . V = 0$

and $u_x = V_y$

$u_y = -V_x$

as $V_y = u_x = 3x^2 - 3y^2 + 1$

$V_x = -u_y = 6xy$

$$\int V_y \, dy = \int (3x^2 - 3y^2 + 1) \, dy$$

$$= 3x^2 y - y^3 + y + \phi_1(x)$$

here $\phi_1(x)$ is the constant term wrt y but can depend on x

$V(x,y) = 3x^2 y - y^3 + y + \phi_1(x)$ —— ①

similary

$V_x = 6xy$

$\Rightarrow \int V_x \, dx = \int 6xy \, dx$

$V(x,y) = 3x^2 y + \phi_2(y)$ —— ②

here ϕ_2 only depends on y as integrated wrt x so const over x.

now by comparing ① and ② we can find $\phi_1(x)$ and $\phi_2(y)$

$V(x,y) = 3x^2 y - y^3 + y + \phi_1(x)$ —— ①

$V(x,y) = 3x^2 y + \phi_2(y)$ —— ②

by comparing ① and ② we get

that $\phi_2(y) = -y^3 + y + c$ ($\because$ Both ① and ② are same)

$\phi_1(x) = c$

$\phi_1(x)$ to be chosen as c as
$v(x,y) = 3x^2y + \phi_2(y)$ or only $3x^2y$ term
in v depends on x
similarly, $\phi_2(y) = -y^3 + y + c$
as
$v(x,y) = 3x^2y - y^3 + y + \phi_1(x)$
or only $3x^2y - y^3 + y$ depends on y in $v(x,y)$

$\Rightarrow v(x,y) = 3x^2y - y^3 + y + c$
where $c \in \mathbb{R}$

now, lets check if v is harmonic or not:

$$V_x = \frac{\partial}{\partial x}(3x^2y - y^3 + y + c)$$
$$= 6xy$$
$$V_{xx} = 6y$$

$$V_y = \frac{\partial}{\partial y}\left(3x^2y - y^3 + y + c\right)$$
$$= 3x^2 - 3y^2 + 1$$
$$V_{yy} = -6y$$

now as V_{xx}, V_{yy} are cont and $V_{xx} + V_{yy} = 0$

$\Rightarrow V \in C^2(\mathbb{R}^2)$ and $\Delta . V = 0$ (V_{xx}, V_{yy} are cont as they are polynomials)

$\Rightarrow v$ is harmonic

also $V_x = 6xy = -U_y$
$V_y = 3x^2 - 3y^2 + 1 = U_x$

we have $U_x = V_y$
$U_y = -V_x$

$\therefore$ C-R equations also satisfied

so $V(x,y) = 3x^2y - y^3 + y + c$ for any $c \in \mathbb{R}$
is the
harmonic conjugate for $u(x,y)$ (given)

2. $$\int_{|z|=1} \frac{\sin(z)}{z^8}\,dz$$

$f(z) = \sin(z)$
then

$$f(z) = \frac{e^{iz} - e^{-iz}}{2i} \quad (\text{By defination of } \sin(z))$$

$$f(x,y) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \frac{e^{ix-y} - e^{-ix+y}}{2i}$$

$$= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i}$$

$$= \frac{e^{-y}}{2}\sin x + \frac{e^{y}}{2}\sin x + \frac{1}{2i}e^{-y}\cos x - \frac{e^{y}}{2i}\cos x$$

$$= \frac{\sin x}{2}\left(e^{y}+e^{-y}\right) + i\,\frac{\cos x}{2}\left(e^{y}-e^{-y}\right)$$

now $f(x,y) = u(x,y) + i v(x,y)$

then

$$u(x,y) = \frac{\sin x}{2}\left(e^{y}+e^{-y}\right)$$

$$v(x,y) = \frac{\cos x}{2}\left(e^{y}-e^{-y}\right)$$

$$u_x = \frac{\cos x}{2}\left(e^{y}+e^{-y}\right) \qquad v_x = -\frac{\sin x}{2}\left(e^{y}-e^{-y}\right)$$

$$u_y = \frac{\sin x}{2}\left(e^{y}-e^{-y}\right) \qquad v_y = \frac{\cos x}{2}\left(e^{y}+e^{-y}\right)$$

then we can observe that

$$\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\} \text{this is true for all } \mathbb{C}$$

also as $\cos x, e^{y}, e^{-y}, \sin x$ is cont and diff (on $\mathbb{C}$)

$\Rightarrow u_x, u_y, v_x, v_y$ is cont and diff (on $\mathbb{C}$)

(composition of functions)

and so $f(z) = \sin(z)$ is holomorphic function over $\mathbb{C}$

then we can use the Cauchy's integral formula for

$C \subseteq \Omega = \mathbb{C}$

$\hookrightarrow C$ is s.t $|z| \leq 1 \;\; \forall z \in C$

i.e $C = \{z \mid |z| \leq 1\}$ then

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\delta C} \frac{f(\omega)}{(\omega - z)^{n+1}}\,d\omega \qquad \text{for } z \in \operatorname{int}(C)$$

$\uparrow$ Boundary of C

δC is a countercloctwise curve

now putting $f(z) = \sin(z)$

we get:

$$f^{(7)}(z) = \frac{7!}{2\pi i}\int_{|\omega|=1} \frac{\sin(\omega)}{(\omega - z)^{8}}\,d\omega$$

$z \in \operatorname{int}(C)$

for $0 = z$

as $0 \in \operatorname{int}(C)$ $\left(\because \text{ for } \delta = \frac{1}{2} \;\; B_{\frac{1}{2}}(0) \subseteq C\right.$

as there is an open ball around 0 which is inside C.

as $\forall z' \in B_{\frac{1}{2}}(0) \Rightarrow |z' - 0| < \frac{1}{2}$

$\left.\Rightarrow z' \in C\right)$

$$f^{(7)}(0) = \frac{7!}{2\pi i} \int_{|w|=1} \frac{\sin(w)}{(w)^8}\, dw$$

now $f(z) = \sin(z)$
$$= \frac{e^{iz} - e^{-iz}}{2i} \quad \text{(By defination)}$$

$$f'(z) = \frac{ie^{iz} + ie^{-iz}}{2i} \quad \text{(By chain rule)}$$
$$= \frac{e^{iz} + e^{-iz}}{2}$$
$$f'(z) = \cos(z) \quad \text{(By defination)}$$

now similarly $f^{(2)}(z) = \frac{\partial}{\partial z}\left(\frac{e^{iz} + e^{-iz}}{2}\right)$
$$= \frac{ie^{iz} - e^{-iz}i}{2}$$
$$= -\frac{(e^{iz} - e^{-iz})}{2i}$$
$$= -\sin(z)$$

now, $f^{(0)}(z) = \sin(z)$
$f^{(1)}(z) = \cos(z)$
$f^{(2)}(z) = -\sin(z)$

$$f^{(3)}(z) = -\frac{\partial}{\partial z}\sin(z) = -\cos(z) \quad (\text{from } f^{(1)}(z))$$

$$f^{(4)}(z) = -\frac{\partial}{\partial z}\cos(z) = \sin(z) \quad (\text{from } f^{(2)}(z))$$

$$f^{(5)}(z) = \frac{\partial}{\partial z}\sin(z) = f^{(1)}(z)$$
$$f^{(6)}(z) = \frac{\partial}{\partial z} f^{(1)}(z) = f^{(2)}(z)$$
$$f^{(7)}(z) = \frac{\partial}{\partial z} f^{(2)}(z) = f^{(3)}(z)$$

so $f^{(7)}(z) = -\cos(z)$

now $f^{(7)}(0) = -\cos(0)$
$$= -\left[\frac{e^0 + e^0}{2}\right] \quad (\because \text{By defination})$$
$$= -1$$

now $f^{(7)}(0) = -1 = \frac{7!}{2\pi i} \int_{|w|=1} \frac{f(w)}{(w)^8}\, dw$

now representing w as z

$$-\frac{2\pi i}{7!} = \int_{|z|=1} \frac{\sin(z)}{(z)^8}\, dz$$

3. $f : \mathbb{C} \longrightarrow \mathbb{C}$ (entire funtion or holomorphic everywhere)
$|f(z)| \leq A|z|^n \quad \forall z \in \mathbb{C}$
and fixed $A > 0$

To prove : $f(z)$ is a degree n polynomial satisfying $f(0) = 0$

proof : As f is holomorphic everywhere:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\delta C} \frac{f(w)\,dw}{(w-z)^{n+1}}$$

for closed disk $C \subseteq \mathbb{C}$ for $z \in int(C)$ (By cauchy's integral formula)

then for $C = \{ z \mid |z - z_0| \leq R \}$ — disk for z_0 as centre

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \oint_{\delta C} \left| \frac{f(w)}{(w-z_0)^{n+1}} \right| |dw|$$

$$\leq \frac{n!}{\cancel{2\pi}} \times \frac{1}{(R)^{n+\cancel{1}}} \times \cancel{2\pi R} \times \sup_{|w-z_0|=R} |f(w)|$$

($\because 2\pi R$ = path length)

$$\Rightarrow |f^{(n)}(z_0)| \leq \frac{n!}{(R)^n} \times \sup_{|w-z_0|=R} |f(w)|$$

now as $|f(z)| \leq A|z|^n \ \forall z$
for w s.t $|w - z_0| = R$
$\Rightarrow |w| \leq R + |z_0| \quad \forall w$ s.t $|w-z_0|=R$
($\because$ triangle inequality)

$$\Rightarrow \sup_{|w-z_0|=R} |f(w)| \leq \sup_{|w-z_0|=R} A|w|^n$$

$$\leq A||z_0| + R|^n$$

$$\text{then } |f^{r}(z_0)| \leq \frac{(r)!}{(R)^r} A||z_0| + R|^n$$

(Here we changed n to r as $A|z|^n$, n is different)

now for $r > n$ we have

$$|f^{r}(z_0)| \leq \frac{(r)!}{(R)^r} A\,||z_0| + R|^n$$

for $R > |z_0|$ we have ($\because$ we can choose R)

$$|f^{r}(z_0)| \leq \frac{(r)!}{(R)^r} A\,||z_0| + R|^n$$

$$\leq \frac{(r)!}{(R)^r} A\,|2R|^n$$

$$= (r)!\, 2^n A \times \frac{1}{(R)^{r-n}}$$

now as $R \to \infty$ and $r-n > 0$
then $\frac{1}{(R)^{r-n}} \to 0$

as $r-n > 0$
and so for $R > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{r-n}}$
and $R > |z_0|$
we have

$$\left| \frac{1}{R^{r-n}} - 0 \right| < \varepsilon$$
$$\forall \varepsilon > 0$$

$$\Rightarrow \lim_{R \to \infty} \frac{1}{(R)^{r-n}} = 0$$

and so $|f^r(z_0)| \leq \lim_{R \to \infty} \frac{r!}{(R)^{r-n}} (2^n) A$
$= 0$
(as this was true for all $R > 0$)

$\Rightarrow |f^r(z_0)| \leq 0$
and $|f^r(z_0)| \geq 0$ (as absolute value)
$\Rightarrow |f^r(z_0)| = 0$
$\Rightarrow f^r(z_0) = 0$ (from property of absolute value)

$\forall r > n \Rightarrow f^{(r)}(z_0) = 0 \quad \forall z_0 \in \mathbb{C}$
(as fuction is holomorphic)

as f is holomorphic $\forall z_0 \in \mathbb{C}$

$$\Rightarrow f(z) = \sum_{r \geq 0} \frac{f^{(r)}(z_0)(z-z_0)^r}{(r)!}$$ (defination of being holomorphic)

for $r > n \Rightarrow f^{(r)}(z_0) = 0$
so

$$f(z) = \sum_{r=0}^{n} \frac{f^{(r)}(z_0)(z-z_0)^r}{r!}$$

$$= f(z_0) + \frac{f'(z_0)(z-z_0)^1}{1!} + \dots \frac{f^{(n)}(z_0)(z-z_0)^n}{n!}$$

now let $z_0 = 0$ then
as $|f(z)| \leq A|z|^n$
$\Rightarrow |f(0)| \leq A(0) = 0$
$\Rightarrow |f(0)| = 0$
$\Rightarrow f(0) = 0$ ($\because$ property of absolute value)

for $z_0 = 0$
then $f(z) = 0 + \frac{f'(0)(z)}{1!} + \frac{f^{(2)}(0)(z)^2}{2!} + \dots + \frac{f^{(n)}(0)(z)^n}{n!}$

let $a_i = \frac{f^{(i)}(0)}{(i)!}$

$$f(z) = a_1 z + a_2 z^2 + \cdots a_n z^n$$

so $f(z)$ is a polynomial of atmost n degree.

wue $f(0) = 0$

(Note : Question said that we have to prove f has degree n but for $f(z)=0$, $|f(z)| \leq A|z|^n$, f is hol on $\mathbb{C}$ but still it does not have degree n)

Tutorial- 9 :

9.1 Residue of $f(z)=\frac{\sin z}{z}$ at $z=0$

now, as $\sin z$ is hol on $\mathbb{C}$
z is hol on $\mathbb{C}$

$\Rightarrow \frac{\sin z}{z}$ is hol on $\mathbb{C}\setminus\{0\}$ ($\because z=0$ is denominator 0)

so $f(z)=\frac{\sin z}{z}$ is hol on $\mathbb{C}\setminus\{0\}$

now $\tilde{f}(z)= \begin{cases} 0 & ; z=0 \\ \frac{z}{\sin z} & ; z\neq 0 \end{cases}$

now $\tilde{f}(z)$ is s.t

for $z=0$: $\lim_{h\to 0} \frac{\tilde{f}(h)-\tilde{f}(0)}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}}{\sin h \cdot \cancel{h}}$

$= \lim_{h\to 0} \frac{1}{\sin h}$ this is not well defined as for $h=\frac{2\pi}{n}$ form different answer then $h=\frac{\pi/2}{n}$

so $\tilde{f}(h)$ is not hol

$\therefore f$ does not have pole at $z_0=0$

$\Rightarrow Res(f,0)=0$

$a_{-1}=0$

now

$f(z)=\frac{\sin z}{z}$

$= \frac{z-\frac{z^3}{3!}+\frac{z^5}{5!}\cdots}{z}$ (taylor series expansion)

$= 1-\frac{z^2}{3!}+\frac{z^4}{5!}\cdots$ (taylor series expansion)

but as no pole at $z=0$

res$(f,0)=0$

from the expansion also that is clear

9.2 $f(z)=\frac{\sin(z)}{z^2}$ at $z=0$

now, $\tilde{f}(z)= \begin{cases} 0 & ; z=0 \\ \frac{z^2}{\sin z} & ; z\neq 0 \end{cases}$

now $\tilde{f}(z)$ is s.t for $z\neq 0$ i.e $\forall z\in\mathbb{C}\setminus\{0\}$

$\tilde{f}(z)=\frac{z^2}{\sin z}$ and is hol ($\because z^2, \sin z$ hol and $\sin z\neq 0$) on a small nbd U around 0 (where $\sin z\neq 0$ except $z=0$)

$\tilde{f}(z)= \begin{cases} 0 & ; z=0 \\ z^2/\sin z & ; z\neq 0 \end{cases}$

now $(\tilde{f}(0))' = \lim_{\substack{h \in \mathbb{C} \\ h \to 0}} \dfrac{\tilde{f}(h) - \tilde{f}(0)}{h}$

$= \lim_{h \to 0} \dfrac{h}{\sin h} = \lim_{h \to 0} \dfrac{2ih}{e^{ih} - e^{-ih}}$

now $\sin h = \dfrac{e^{ih} - e^{-ih}}{2i}$

$= \lim_{h \to 0} \dfrac{2ih\, e^{ih}}{e^{2ih} - 1}$

$= \lim_{h \to 0} \dfrac{2ih\, e^{ih}}{e^{2ih} - 1}$

$= \lim_{h \to 0} \dfrac{2ih e^{ih}}{\cancel{1} + (2ih) + \frac{(2ih)^2}{2!} + \frac{(2ih)^3}{3!} \dots - \cancel{1}}$

$= \lim_{h \to 0} \dfrac{e^{ih}}{1 + \frac{2ih}{2!} + \frac{(2ih)^2}{3!} + \dots}$

$= 1$

so $\tilde{f}'(0)$ exist and is 1

$\therefore \tilde{f}$ is hol ($\forall z \in \mathbb{C}$, $f'(z)$ is well defined on the small nbd)

now, this mean that f has a pole at $z = 0$ then by residue formula over on the small nbd disk

$D = \{ z \in \mathbb{C} \mid |z - 0| < \frac{1}{n} \}$

well $0 \in \mathrm{int}(D)$ f is hol on small nbd
$\Rightarrow f$ is hol on $D \setminus \{0\}$

so $\mathrm{Res}(f, 0) = \dfrac{1}{2\pi i} \oint_{\delta D} f(z)\, dz$

$= \dfrac{1}{2\pi i} \oint_{\delta D} \dfrac{\sin z}{z^2}\, dz$

$= \dfrac{1}{2\pi i} \oint_{\delta D} \dfrac{\sin z}{(z - 0)^2}\, dz$

as $g(z) = \sin z$ then

$g'(z_0) = \dfrac{1}{2\pi i} \oint_{\delta D} \dfrac{\sin(z)}{(z - z_0)^2}\, dz$

from cauchy riemann equation for $z_0 = 0$

$g'(0) = \cos(0)$
$= 1 = \dfrac{1}{2\pi i} \oint_{\delta D} \dfrac{\sin z}{z^2}\, dz$

$\mathrm{Res}(f, 0) = 1$

9.3 $f(z) = \frac{1}{e^z - 1} - \frac{1}{z}$

now as $\frac{1}{e^z-1} - \frac{1}{z}$ is hol on $\mathbb{C}\setminus\{0\}$

($\because$ $e^z - 1, z$ is hol and $e^z - 1 \neq 0$, $z \neq 0$ for $z \neq 0$)

then $g(z) = \begin{cases} -1/2 & ; z = 0 \\ \frac{1}{e^z-1} - \frac{1}{z} & ; z \neq 0 \end{cases}$

now $g(z)$ is s.t $\forall z \in \mathbb{C}\setminus\{0\}$ it is hol

and for $z=0$:

$$g'(0) = \lim_{h\to 0} \frac{g(z) - g(0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{e^h-1} - \frac{1}{h} + \frac{1}{2}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{e^h-1} - \frac{1}{h} + \frac{1}{2}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{e^h-1} - \frac{2}{2h} + \frac{h}{2h}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{e^h-1} + \frac{(h-2)}{2h}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{2h + (h-2)(e^h-1)}{2h(e^h-1)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\cancel{2h} + he^h - \cancel{h} - 2e^h + 2}{2h(e^h-1)}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\cancel{h} + h(\cancel{1} + \cancel{h} + h^2/2! + \cdots) - 2(\cancel{1} + \cancel{h} + \cancel{h^2/2!} \cdots) + \cancel{2}}{2h(\cancel{1} + h + \frac{h^2}{2!} + \cdots - \cancel{1})}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{h(h^2/2! + h^3/3! + \cdots) - 2(h^3/3! + \cdots)}{2h(h + h^2/2! + \cdots)}\right]$$

$$= \lim_{h\to 0} \left[\frac{\frac{1}{2!} + \frac{h}{3!} + \frac{h^2}{4!} + \cdots - 2\left(\frac{h}{4!} + \frac{h^2}{5!} + \cdots\right)}{2\left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \cdots\right)}\right]$$

$$= \frac{1}{4}$$

so $g'(z)$ exist $\forall z \in \mathbb{C}$

$\Rightarrow g(z)$ is hol

so $\exists g$ s.t $g: \mathbb{C} \to \mathbb{C}$

s.t $g(z) = f(z)$ $\forall z \in \mathbb{C}\setminus\{0\}$

and g is hol $\forall z \in \mathbb{C}$

or use the theorm Riemanns theorm to get $\lim_{z\to z_0} (z - z_0) f(z) = 0$ $\Rightarrow z_0$ is removabl

$\therefore$ $Z=0$ is a removable singularity

9.4 meromorphic fuction

$$f(z)=\frac{\pi^2}{\sin^2(\pi z)}$$

now $\sin(z)=0$ for what points in $\mathbb{C}$ lets see

now, $\sin(z)=0$

then $\frac{e^{iz}-e^{-iz}}{2i}=0$

$\Rightarrow e^{i(x+iy)}-e^{-i(x+iy)}=0$

$\Rightarrow e^{ix-y}-e^{-ix+y}=0$

$\Rightarrow e^{ix-y}=e^{-ix+y}$

$\Rightarrow e^{-y}(\cos(x)+i\sin(x))$
$=e^{y}(\cos(-x)+i\sin(-x))$

$\Rightarrow e^{-y}=e^{y}$ & $\sin(x)=\sin(-x)$

so $y=0$ and $\sin(x)=\sin(-x)$
$\Rightarrow \sin(x)=0$
$\Rightarrow x=0,\pi,2\pi,3\pi\ldots$

$\therefore \sin(z)=0$ for $z=n\pi$ $\forall n\in\mathbb{Z}$

now $f(z)=\frac{\pi^2}{\sin^2(\pi z)}$ is not defined for $z=n$ $\forall n\in\mathbb{Z}$ $\left(\because \sin(z)=0 \Leftrightarrow z=n\pi\right)$

$\therefore$ singuarites (or isolated singuaries) of this fuction are $z=n$ $\forall n\in\mathbb{Z}$

also as given meromorphic, they are poles of the fuction by defination.

There will be removable singueries at all $z\in\mathbb{C}\setminus\{n\pi\}$ $\forall n\in\mathbb{Z}$

9.5 $f(z)=\frac{\pi^2}{\sin^2(\pi z)}$

firstly for some $n\in\mathbb{Z}$

$$\tilde{f}_n(z)=\begin{cases} 0 & ; z=n \\ \frac{\sin^2(z\pi)}{\pi^2} & ; \underbrace{|z-n|<1}_{D\setminus\{n\}\text{ some nbd}}, z\neq n \end{cases}$$

then as $\sin^2(z\pi)$ is holo
$\tilde{f}_n(z)$ is holo on $D\setminus\{n\}$

also $\lim_{h\to 0}\frac{\tilde{f}_n(n+h)-\tilde{f}_n(n)}{h}=\lim_{h\to 0}\frac{\sin^2(\pi n+\pi h)}{\pi^2 h}$

$=\lim_{h\to 0}\frac{[\cancel{\sin(\pi n)\cos(\pi h)}+\cos(\pi n)\sin(\pi h)]^2}{\pi^2 h}$

$$= \lim_{h\to 0} \frac{\sin^2(\pi h)}{\pi^2 h}$$

$$= \lim_{h\to 0} \frac{1}{\pi} \frac{\sin^2(h)}{h}$$

$$= \lim_{h\to 0} \frac{1}{\pi} \times \frac{1}{2i} \times \frac{e^{ih} - e^{-ih}}{h}$$

$$= \lim_{h\to 0} \frac{1}{\pi} \times \frac{1}{2i} \times \frac{\left[\cancel{1} + ih + \frac{(ih)^2}{2!} + \dots - \cancel{1} - (-ih) - \frac{(ih)^2}{2!} \dots\right]}{h}$$

$$= \lim_{h\to 0} \frac{1}{\pi} \times \frac{1}{\cancel{2i}} \times \frac{\cancel{2ih}}{\cancel{h}}$$

$$= \frac{1}{\pi}$$

so $\tilde{f}_n(z)$ is hol on D

$\therefore$ $z = n$ is pole of f

now as $\tilde{f}_n(z) : D \longrightarrow \mathbb{C}$ is s.t

it is hol

and it is zero at $z = n$

then

on D: as $\tilde{f}_n(z) = \frac{\sin^2(z\pi)}{\pi^2}$

$$\tilde{f}_n(n) = 0$$

$$\tilde{f}_n'(z) = 2\sin(z\pi)\cos(z\pi)\frac{\cancel{\pi}}{\pi^{\cancel{2}}}$$

$$= \frac{\sin(2z\pi)}{\pi}$$

$$\tilde{f}'(n) = 0$$

$$\tilde{f}_n''(n) = \cos(2\pi n) \cdot 2$$

$$= 2 = (2)(2\pi)^0(-1)^0$$

$$\tilde{f}_n'''(n) = 2(2\pi)(-\sin(2\pi z)) = 0$$

$$\tilde{f}_n^{4}(n) = 2(2\pi)(2\pi)(-1)(\cos 2\pi n)$$

$$= 2(2\pi)^2(-1)^1$$

$$\tilde{f}_n^{6}(n) = 2(2\pi)^4$$

$$\tilde{f}_n^{2k}(n) = 2(2\pi)^{2k-2}(-1)^{k+1}$$

$k = 1, 2, 3 \dots$

now $$\tilde{f}_n(z) = \sum_{k=1}^{\infty} (z-n)^{2k}(2)(2\pi)^{2k-2}(-1)^{k+1}$$

$h_n(z)$ non zero D

$$= (z-n)^2 \left[\sum_{k=1}^{\infty} (z-n)^{2k-2}(2)(2\pi)^{2k-2}(-1)^{k+1}\right]$$

so $f(z) = (z-n)^{-2}\left[\sum_{k=1}^{\infty}(z-n)^{2k-2}(2)(2\pi)^{2k-2}(-1)^{k+1}\right]^{-1}$

$$\tilde{f}_n(z) = \left(\frac{\frac{e^{i(\pi z)}-e^{-i(\pi z)}}{2i}}{\pi^2}\right)^2$$

$$= \frac{1}{-4} \times \frac{1}{\pi^2} \times \left[e^{2\pi z i} + e^{-2\pi z i} - 2\right]$$

$$= -\frac{1}{4}\frac{1}{\pi^2}\left[e^{2\pi z i} + e^{-2\pi z i} - 2\right]$$

→ doubt

now $f(z) = (z-n)^2 \times \frac{1}{(z-n)^2} \times \frac{(-4\pi^2)}{(e^{2\pi z i} + e^{-2\pi i z} - 2)}$

$$= (z-n)^{-2}\left[\frac{(z-n)^2(-4\pi^2)}{e^{2\pi z i} + e^{-2\pi z i} - 2}\right]$$

compute the principle part of $f(z) = \frac{\pi^2}{\sin^2(\pi z)}$ $z = n$ for $n \in \mathbb{Z}$

$$= \frac{\pi^2}{\sin(\pi z - \pi n)^2}$$

$$= \frac{\pi^2}{\left((\pi z - \pi n) - \frac{(\pi z - \pi n)^3}{3!} + \cdots\right)^2} \to \frac{1}{(z-n)^2} + \cdots$$

this method

more generally $\lim_{z \to n} \frac{\sin(\pi z)}{z-n} = \frac{d}{dz}\sin(\pi z)\Big|_{z=n} = \pi\cos(nz) = \pm\pi$

thus at $z = n$ is a zero of order 1 of $\sin(\pi z)$

$\Rightarrow z = n$ is a pole of order 1 of $\frac{1}{\sin(\pi z)}$

$\Rightarrow z = n$ is a pole of order 2 of $\frac{\pi^2}{\sin^2(\pi z)}$

Thus, $f(z) = \underbrace{\frac{a_{-2}}{(z-n)^2} + \frac{a_{-1}}{(z-n)}}_{\text{principle part}} + a_0 + \cdots$

now $(z-n)^2 f(z) = a_{-2} + a_{-1}(z-n) + \cdots$

$$\lim_{z \to n} (z-n)^2 f(z) = a_{-2}$$

$$\lim_{z \to n} \frac{d}{dz}(z-n)^2 f(z) = a_{-1}$$

now $\lim_{z \to n} (z-n)^2 \times \frac{\pi^2}{[\sin(\pi z)^2 - \sin(\pi n)^2]}$

$$= \lim_{z \to n}\left(\frac{\pi(z-n)}{\sin(\pi z)}\right)^2$$

$$= \lim_{z \to n}\left(\frac{\pi z - \pi n}{\sin(\pi z - \pi n)}\right)^2$$

$$= \frac{1}{\cos(\pi n)^2} = 1$$

$$a_{-1} = \lim_{z \to n}\frac{d}{dz}\left[\frac{(z-n)^2\pi^2}{\sin^2(\pi z)}\right] = \pi^2 \lim_{z \to n}(z-n)\left[\frac{2\sin(\pi z) - 2\pi(z-n)\cos(\pi z)}{\sin^3 \pi z}\right]$$

$= 0$ using taylor expansion

so $\frac{1}{(z-i)^2}$ is principle part

Tutorial-10:

10.1 $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$

now let $I = \int_{-\infty}^{\infty} e^{-\pi x^2} dx$

$$= \int_{-\infty}^{\infty} e^{-\pi y^2} dx$$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-\pi x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-\pi y^2} dy\right)$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-\pi y^2} dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)} dx\, dy$$

$\mathbb{R}^2 \longrightarrow (r,\theta)$

now $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\}$ change of variables $\quad g(x,y) = (r\cos\theta, r\sin\theta)$

then

$$J = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$ Jacobian matrix

diffeomorphism

$|\det J| = r$

$r > 0$, now $I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)} dx\, dy$

$$= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-\pi r^2}(r)\, dr\, d\theta$$

let $-e^{-\pi r^2} = t$

$dt = e^{-\pi r^2}(+2\pi r)$

$\Rightarrow -\frac{dt}{2\pi} = e^{-\pi r^2} dr$

$$= \int_{0}^{2\pi}\int_{-1}^{0} -\frac{dt}{2\pi}\, d\theta$$

$$= \int_{0}^{2\pi}\int_{0}^{1} \frac{dt}{2\pi}\, d\theta$$

$$= \frac{1}{2\pi} \times 1 \times \int_{0}^{2\pi} d\theta$$

$$= \frac{2\pi}{2\pi} = 1$$

so $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$

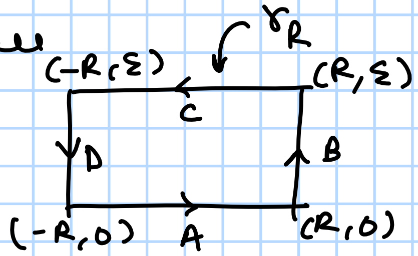
10.2 $f(x)$ is its own fourier transformation

so for $\xi \in \mathbb{R}$

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i \xi x} dx = e^{-\pi \xi^2}$$

now wlog $\varepsilon > 0$, then

$f(Z) = e^{-\pi Z^2}$



now $f'(Z) = e^{-\pi Z^2}(-\pi \times 2Z)$
$= \underbrace{-2\pi Z e^{-\pi Z^2}}_{\text{exist } \forall Z \in \mathbb{C}}$

$\Rightarrow f(Z)$ is holomorphic $\forall Z \in \mathbb{C}$

so it does not have any poles

now, so By cauchys theorm

$$\int_{\gamma_R} f(Z) = 0 = \int_A f(Z) + \int_B f(Z) + \int_C f(Z) + \int_D f(Z)$$

now $\int_A f(Z) = \int_{-R}^{R} e^{-\pi x^2} dx$ for $\gamma(t) = t$, $t \in [-R, R]$

$$\int_B f(Z)\,dZ = \int_0^{\varepsilon} e^{-\pi(R+it)^2}(i)\,dt$$ for $\gamma(t) = R + it$, $t \in [0,\varepsilon]$

$$= \int_0^{\varepsilon} e^{-\pi(R^2 - t^2 + 2\pi i t R)}(i)\,dt$$

$$\left|\int_B f(Z)\,dZ\right| \leq \left|\int_0^{\varepsilon} e^{-\pi(R^2 - t^2 + 2\pi i t R)}\,dt\right|$$

$$\leq e^{-\pi R^2}\left|\int_0^{\varepsilon} e^{\pi t^2}\,dt\right|$$

$= c$ (some const not dependent on R)

so $\left|\int_B f(Z)\,dZ\right| \leq c e^{-\pi R^2} \to 0$ as $R \to \infty$ $= c\, e^{-\pi R^2}$

so $\int_B f(Z)\,dZ \to 0$ as $R \to \infty$

now $$\int_D f(Z)\,dZ = \int_{\varepsilon}^{0} e^{-\pi(-R+it)^2}(i)\,dt$$ $t \in [0,\varepsilon]$

$$= \int_0^{\varepsilon} e^{-\pi(-R-it)^2}(i)(-dt)$$

$$\left|\int_D f(z)\right| \leqslant e^{-\pi R^2}\left|\int_0^{\varepsilon} e^{-2\pi i t R+\pi t^2}\,dt\right|$$

$$\leqslant e^{-\pi R^2} C$$

so as $R \longrightarrow \infty$

$$\int_D f(z)\,dz \longrightarrow 0$$

now for $R \longrightarrow \infty$

$$\lim_{R\to\infty} \int_A f(z)\,dz + \int_B f(z)\,dz + \int_C f(z)\,dz + \int_D f(z)\,dz = 0$$

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx + 0 + \lim_{R\to\infty} \int_R^{-R} e^{-\pi(t+i\varepsilon)^2}\,dt = 0 \quad \text{for } t\in[-R,R]$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx - \int_{-\infty}^{\infty} e^{-\pi(t+i\varepsilon)^2}\,dt = 0$$

$$\Rightarrow 1 - \int_{-\infty}^{\infty} e^{-\pi(x^2-\varepsilon^2+2xi\varepsilon)}\,dx = 0$$

$$\Rightarrow 1 = e^{\pi\varepsilon^2}\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi x i\varepsilon}\,dx$$

$$\Rightarrow e^{-\pi\varepsilon^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi x i \varepsilon}\,dx$$

similar for $\varepsilon<0$, $\varepsilon=0$ ↳ Already proved case in previous Question

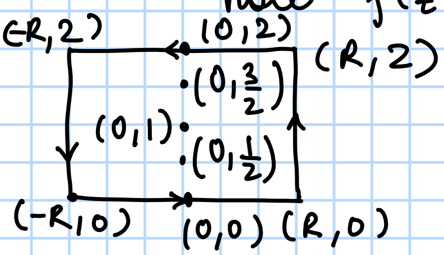
10.3 $\cosh(z) = \dfrac{e^z+e^{-z}}{2}$

$f(x) = \dfrac{1}{\cosh \pi x}$

now let $f(z) = \dfrac{e^{-2\pi i\varepsilon z}}{\cosh(\pi z)}$

To show: $\forall \varepsilon \in \mathbb{R}$

$$\int_{-\infty}^{\infty} \frac{1}{\cosh \pi x} e^{-2\pi i \varepsilon x}\,dx = \frac{1}{\cosh \pi \varepsilon}$$



now $f(z)$ has poles when $\cosh \pi z = 0$

$\Rightarrow e^{\pi z} = -e^{-\pi z}$

$\Rightarrow e^{2\pi z} = -1$

as $e^{i\pi} = -1$

we have

$i\cancel{\pi} + 2\cancel{\pi} n i = 2\cancel{\pi} z$

$\Rightarrow z = \frac{i}{2} + ni$ ⌉ $\forall n \in \mathbb{N}$ these are poles

now as $\dfrac{1}{f(z)} = \cosh(\pi z) \times e^{2\pi i \varepsilon z}$

$$\frac{1}{f(z)} = \left(\frac{e^{\pi z}+e^{-\pi z}}{2}\right) \times e^{2\pi i s z}$$

now for z around $i/2$
we have

$$\tilde{f}(z) = \begin{cases} 0 & ; z = i/2 \\ \frac{1}{f(z)} & ; \text{otherwise} \end{cases}$$

now as $\frac{1}{f(z)} = \frac{1}{e^{\pi z} \times 2}\left[e^{2\pi z}+1\right] g(n)$

$$= \tilde{g}(n)\left[e^{2\pi z}+1\right]$$

$$= \tilde{g}(n)\left[\frac{e^{2\pi z-\pi i}}{e^{-\pi i}}+1\right]$$

$$= \tilde{g}(n)\left[\cancel{-1}-(2\pi z-\pi i)-\frac{(2\pi z-\pi i)^2}{2!}+\dots \;\cancel{+1}\right]$$

$$= (2\pi z-\pi i)\, G(n)$$

holomorphic

so $\alpha = i/2$ is a simple zero for $1/f(z)$
$\Rightarrow \alpha = i/2$ is a simple pole for $f(z)$

now, then $n=1$
we get

$$\operatorname{Res}(f,\alpha) = \lim_{z\to\alpha} (z-\alpha) f(z)$$

$$= \lim_{z\to i/2} \left(z-\frac{i}{2}\right)\frac{e^{-2\pi i s z}}{\cosh(\pi z)}$$

$$= \lim_{z\to \frac{i}{2}} \left(z-\frac{i}{2}\right)\frac{2\times e^{-2\pi i s z}}{e^{\pi z}+e^{-\pi z}}$$

$$= \lim_{z\to i/2} \frac{\left(z-\frac{i}{2}\right)\times 2\times e^{\pi z}\times e^{-2\pi i s z}}{e^{2\pi z}-e^{\pi i}}$$

$$= \lim_{2\pi z\to \pi i} \frac{(2\pi z-\pi i)\times 2\times e^{\pi z}\times e^{-2\pi i s z}}{2\pi\,(e^{2\pi z}-e^{\pi i})}$$

$$= \frac{2\times e^{\pi(i/2)}\times e^{-2\pi i s(i/2)}}{2\pi \times e^{2\pi(i/2)}}$$

$$= \frac{\cancel{2}[i]\times e^{\pi s}}{\cancel{2}\pi\times(-1)}$$

$$= \frac{e^{\pi s}}{\pi i}$$

similarly $\operatorname{Res}\left(f, \underset{\beta_{11}}{\tfrac{3}{2}i}\right) = \lim_{z\to\frac{3}{2}i} \left(z-\frac{3}{2}i\right) f(z)$

will also have $n=1$

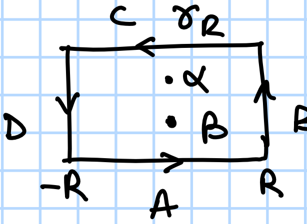
$$= \lim_{z \to \frac{3}{2}i} \left(z - \tfrac{3}{2}i\right) \times \frac{e^{-2\pi i \varepsilon z} \times 2}{e^{\pi z} + e^{-\pi z}}$$

$$= \lim_{z \to \frac{3}{2}i} \frac{\left(z - \frac{3}{2}i\right) \times e^{\pi z} \times 2 \times e^{-2\pi i \varepsilon z}}{e^{2\pi z} - e^{(3/2 i)2\pi}}$$

$$= \frac{e^{\pi(3/2 i)} \times \cancel{2} \times e^{-2\pi i \varepsilon (3/2 i)}}{\cancel{2}\pi \; e^{(3/2 i)(2\pi)}}$$

$$= \left[i(+1)\right] \times e^{3\pi\varepsilon} \times \frac{1}{\pi \times (+1)}$$

$$= -\frac{e^{3\pi\varepsilon}}{\pi i}$$

now 

then $\int_{\gamma_R} f(z) = \left(\operatorname{Res}(f,\alpha) + \operatorname{Res}(f,\beta)\right) 2\pi i$

where $\int_B f(z)\,dz = \int_0^2 \frac{e^{-2\pi i \varepsilon (R+ti)}}{\cosh(\pi)(R+ti)} (i)\,dt$

$\gamma(t) = R + ti$
$t \in [0,2]$ where $|e^{-2\pi i \varepsilon (R+ti)}|$

$\leq e^{4\pi|\varepsilon|}$

and $|\cosh \pi z| = \left|\frac{e^{\pi z} + e^{\pi z}}{2}\right| \geq \frac{1}{2}\left||e^{\pi z}| - |e^{\pi z}|\right|$

$\geq \frac{1}{2}|e^{\pi R} - e^{-\pi R}| \longrightarrow \infty$

as $R \to \infty$

so $\int_B f(z)\,dz \longrightarrow 0$ as $R \longrightarrow \infty$

now for D: $\int_2^0 f(-R+it)(i)\,dt$

$$= -\int_0^2 f(-R+it)(i)\,dt$$

$$\left|\int_0^2 f(-R+it)\,dt\right| \leq \frac{e^{4\pi|\varepsilon|} \times (2)}{\frac{1}{2}|e^{-\pi R} - e^{\pi R}|} \longrightarrow 0 \text{ as } R \longrightarrow \infty$$

so $\int_D f(z)\,dz \longrightarrow 0$ as $R \longrightarrow \infty$

now as $R \to \infty$ $\int_{\gamma_R} f(z)\,dz = \int_A f(z)\,dz + \int_C f(z)\,dz$ $= (\operatorname{Res}(f,\alpha) + \operatorname{Res}(f,\beta))2\pi i$

$$\text{now, } \int_{-\infty}^{\infty} f(x)\,dx - \int_{-\infty}^{\infty} f(x+2i)\,dx = \left(\frac{e^{\pi\varepsilon}}{\pi i} - \frac{e^{3\pi\varepsilon}}{\pi i}\right)2\pi i$$

$$= \int_{-\infty}^{\infty} \frac{1}{\cos(\pi x)} e^{-2\pi i\varepsilon x}\,dx - \int_{-\infty}^{\infty} \frac{e^{-2\pi i\varepsilon x}\, e^{+4\pi\varepsilon}}{\cos(\pi x + 2\pi i)}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\cos(\pi x)} e^{-2\pi i\varepsilon x}\,dx - \int_{-\infty}^{\infty} \frac{e^{-2\pi i\varepsilon x}\, e^{4\pi\varepsilon}}{\cos(\pi x)}\,dx$$

$$I - e^{4\pi\varepsilon} I = \left(\frac{e^{\pi\varepsilon}}{\cancel{\pi i}} - \frac{e^{3\pi\varepsilon}}{\cancel{\pi i}}\right)2\cancel{\pi i}$$

$$I = \frac{2\,(e^{\pi\varepsilon} - e^{3\pi\varepsilon})}{1 - e^{4\pi\varepsilon}}$$

$$= \frac{2\,(e^{-\pi\varepsilon} - e^{\pi\varepsilon})}{(e^{-2\pi\varepsilon} - e^{2\pi\varepsilon})}$$

$$= \frac{2\,(e^{\pi\varepsilon} - e^{-\pi\varepsilon})}{e^{2\pi\varepsilon} - e^{-2\pi\varepsilon}}$$

$$= \frac{2\,\cancel{(e^{\pi\varepsilon} - e^{-\pi\varepsilon})}}{\cancel{(e^{\pi\varepsilon} - e^{-\pi\varepsilon})}\,(e^{\pi\varepsilon} + e^{-\pi\varepsilon})}$$

$$I = \frac{2}{e^{\pi\varepsilon} + e^{-\pi\varepsilon}} = \frac{1}{\cosh \pi\varepsilon}$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{\cos \pi x} e^{-2\pi i\varepsilon x}\,dx = \frac{1}{\cosh \pi\varepsilon}$$

Tutorial - 11 :

11.1 $\Omega \subseteq \mathbb{C}$ open

$D(0,r) \subseteq \Omega$
$\{z \mid |z| \leq r\}$

$f : \Omega \longrightarrow \mathbb{C}$ hol on $D(0,1)$ except $z = 0$

now $\frac{1}{2\pi i} \int_{\delta D(0,1)} f(z)\,dz = \mathrm{Res}(f,0)$

then $\frac{1}{2\pi i} \int_0^{2\pi} f(e^{it})(ie^{it})\,dt = \mathrm{Res}(f,0)$

$\gamma(t) = e^{it}$

now, $f(z) = \frac{\mathrm{Res}(f,0)}{z} + a_0 + a_1 z + a_2 z^2 + \cdots$

then $\gamma(t) = re^{it} \subseteq \delta D(0,r)$

now $\frac{1}{2\pi i} \int_0^{\theta} f(re^{it})(ir)e^{it}\,dt$

$= \frac{1}{2\pi i} \int_0^{\theta} \left[\frac{\mathrm{Res}(f,0)}{re^{it}} + a_0 + a_1 re^{it} + \cdots \right](ir)e^{it}\,dt$

$= \frac{1}{2\pi} \int_0^{\theta} \left[\mathrm{Res}(f,0) + a_0 re^{it} + a_1 r^2 e^{2it} + \cdots \right] dt$

$= \frac{1}{2\pi} \left[\theta \,\mathrm{Res}(f,0) + \left[r a_0 \int_0^{\theta} e^{it} + \cdots \right] \right]$

as $r \longrightarrow 0$

$\lim_{r \to 0} \frac{1}{2\pi i} \int_0^{\theta} f(re^{it})(ir)e^{it}\,dt$

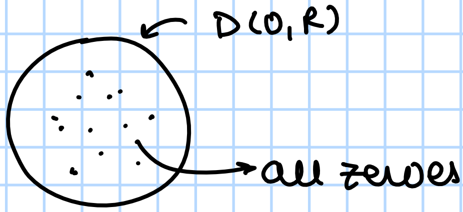
$= \frac{\theta}{2\pi} \mathrm{Res}(f,0) + \underbrace{0 + 0 \cdots}_{\text{as cost term } \times\, r \text{ but } r \longrightarrow 0}$

$\lim_{r \to 0} \frac{1}{2\pi} \int_0^{\theta} f(re^{it})(re^{it})\,dt = \frac{\theta}{2\pi} \mathrm{Res}(f,0)$

11.2 $p(z)$ polynomial of degree n

$p(z) = \prod_{i=1}^{n} (z - a_i)$ where a_i's are roots

$P(z)$ is s.t $a_i \in \text{int}(D(0,R))$ $\forall\, i = 1,2\ldots n$



now $P(z) = \prod_{i=1}^{n} (z - a_i)$

then

$$P'(z) = \sum_{i=1}^{n} \left(\prod_{j \neq i}^{n} (z - a_j) \right)$$

now $$\frac{P'(z)}{P(z)} = \frac{\sum_{i=1}^{n} \left(\prod_{j\neq i}^{n} (z-a_j) \right)}{\prod_{i=1}^{n} (z-a_i)} = \sum_{i=1}^{n} \frac{1}{(z-a_i)}$$

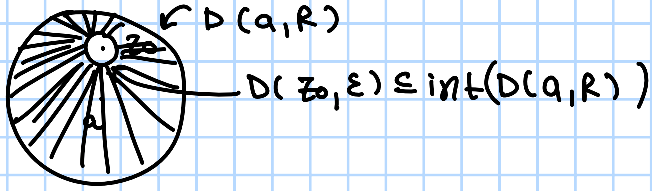
$\text{Res}(g, a_i) = 1 \quad \forall i = 1,2\ldots n$

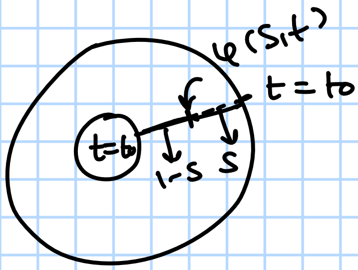
now let $g(z) = \frac{P'(z)}{P(z)}$ then we can see that $g(z)$ has n poles $\{a_i \mid i = 1\ldots n\}$

now $$\frac{1}{2\pi i} \int_{\gamma} \frac{P'(z)}{P(z)}\, dz = \sum_{i=1}^{n} \text{Res}(g(z), a_i)$$
$$= 1 + 1 + \ldots 1$$
$$= n$$

11.3 $D(a,R) = \{ z \mid |z-a| \leq R \}$

$D(z_0, \varepsilon) = \{ z \mid |z - z_0| \leq \varepsilon \}$





now, $\gamma_0(t) = a + Re^{it}$
$\gamma_1(t) = z_0 + \varepsilon e^{it}$
$0 \leq t \leq 2\pi$

now $\varphi(s,t) : [0,1] \times [0,2\pi] \longrightarrow D'(a,R)$

$$D'(a,R) = D(a,R) - \text{int}(D(z_0,\varepsilon))$$

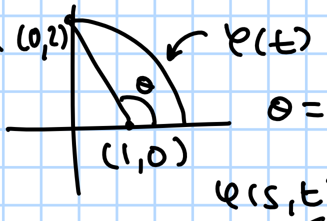
$\varphi(0,t) = \varphi_0(t)$
$\varphi(1,t) = \varphi_1(t)$

$$\varphi(s,t) = (1-s)\varphi_0(t) + s\varphi_1(t)$$

then

$\varphi(s,t)$ is continious as linear combination and we can see graphically it scatisfies given condition

11.4

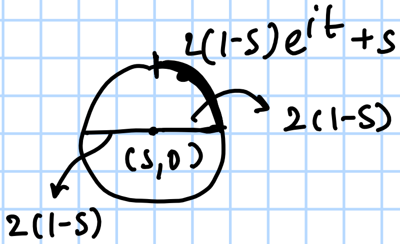


$\theta = \pi - \tan^{-1}(2)$

$\varphi(s,t) = (1-s)2e^{it} + s$
$0 \leq s \leq 1$

$\varphi_s(t) = \varphi(s,t)$
now $\varphi_s(t) = (1-s)2e^{it} + s$

then $\frac{1}{2\pi i}\int_0^{2\pi} f(1+e^{it})(i)(e^{it})\,dt$
$= \mathrm{Res}(f,1)$
$= 1$



$\frac{1}{2\pi i}\int_0^{\pi/2} f(2(1-s)e^{it}+s)(2)(1-s)e^{it}(i)\,dt$

$= \frac{1}{2\pi \cancel{i}}\int_0^{\pi/2} \frac{1 \times 2 \times (1-s)e^{it}\cancel{(i)}\,dt}{2(1-s)e^{it}+s-1}$

$= \frac{1}{2\pi}\int_0^{\pi/2} \frac{2e^{it}}{2e^{it}-1}\,dt$

$2e^{it}-1 = u$
$2e^{it}(i)\,dt = du$

$= \frac{1}{2\pi}\int_1^{2i-1} \frac{1}{i}\frac{du}{u}$

$= \frac{1}{2\pi i}\left(\ln|2i-1| - \ln|1|\right)$

$\ln(2i-1)$
$= \ln \sqrt{5}\, e^{i[\pi - \tan^{-1}2]}$

$\ln(1) = \ln(1 \times e^{2\pi i n})$
$= 2\pi n$
$n=0$ for principle part

$= \frac{1}{2\pi i}\left[\sqrt{5} + i(\pi - \tan^{-1}2)\right]$

$= \frac{\sqrt{5}}{2\pi i} + \frac{\pi - \tan^{-1}(2)}{2\pi}$

Assignment - 3

Dhairya
2383321

1. To prove: $\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi}\,dx = e^{-\pi \xi^2}$ for $\xi \in \mathbb{C}$

proof: First let's show that

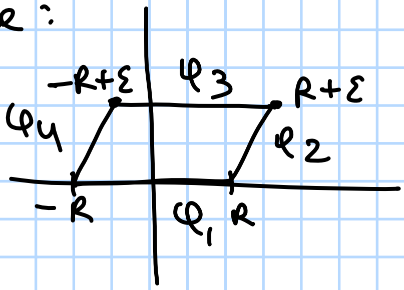
$$f(z) = e^{-\pi z^2} \text{ is hol}$$

as, e^z is hol and $-\pi z^2$ is hol (polynomial is hol)
the composition is also hol
$\Rightarrow f(z)$ is hol $\forall z \in \mathbb{C}$

now then by Cauchy's theorem, any closed piecewise path say φ then

$$\int_\varphi f = 0$$

now let φ be:



where $R \in \mathbb{R}$, $\xi \in \mathbb{C}$ (given)

then $\int_\varphi f = \int_{\varphi_1} f + \int_{\varphi_2} f + \int_{\varphi_3} f + \int_{\varphi_4} f = 0$

where φ is just a line segment

now, $\int_{\varphi_2} f = \int_0^1 f(\varphi_2(t))\,\varphi_2'(t)\,dt$ $\left(\begin{aligned} \varphi_2(t) &= R(1-t) + (R+\xi)(t) \\ &= R + \xi t \\ \varphi_2'(t)\,dt &= \xi\,dt \end{aligned} \right)$

$$= \int_0^1 e^{-\pi[R+\xi t]^2}[\xi\,dt] \quad \text{— ①}$$

similarly $\int_{\varphi_4} f = \int_0^1 f(\varphi_4(t))\,\varphi_4'(t)\,dt$ $\left(\begin{aligned} \varphi_4(t) &= (-R+\xi)(1-t) + (-R)(t) \\ &= -R + \cancel{Rt} + \xi - \xi t - \cancel{Rt} \\ &= -R + \xi - \xi t \\ \varphi_4'(t)\,dt &= -\xi\,dt \end{aligned} \right)$

$$= \int_0^1 e^{-\pi[-R+\xi-\xi t]^2}[-\xi\,dt]$$

$$= \int_0^1 -e^{-\pi[-R+\xi(1-t)]^2}\,\xi\,dt$$

Substitute $u = 1-t$
$du = -dt$

$$= \int_{1-0}^{1-1} +e^{-\pi[-R+\xi u]^2}\,\xi\,(+du)$$

$$= -\int_0^1 e^{-\pi[R-\xi u]^2}\,\xi\,du$$

let $t = u$

$$= -\int_0^1 e^{-\pi[R-\xi t]^2}[\xi\,dt] \quad \text{— ②}$$

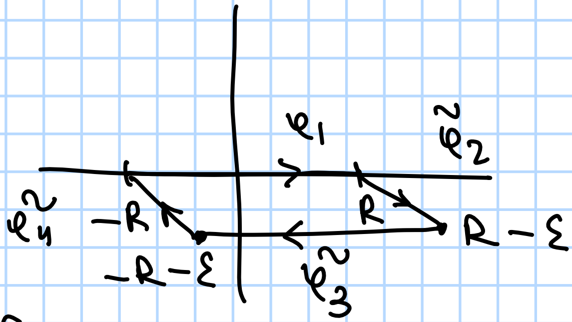
now adding ①, ②

$$\int_{\varphi_2} f + \int_{\varphi_4} f = \int_0^1 e^{-\pi[R+\varepsilon t]^2}\varepsilon\, dt - \int_0^1 e^{-\pi[R-\varepsilon t]^2}\varepsilon\, dt$$

$$= \int_0^1 e^{-\pi[R^2+2\varepsilon tR+\varepsilon^2 t^2]}\varepsilon\, dt - \int_0^1 e^{-\pi[R^2+\varepsilon^2 t^2-2R\varepsilon t]}\varepsilon\, dt$$

$$= \varepsilon e^{-\pi R^2}\int_0^1 e^{-\pi\varepsilon^2 t^2}\left[e^{-\pi 2\varepsilon tR} + e^{+\pi 2\varepsilon tR}\right] dt$$

now let's do the same on a new φ:



then $\displaystyle\int_{\varphi_1} f + \int_{\tilde{\varphi}_2} f + \int_{\tilde{\varphi}_3} f + \int_{\tilde{\varphi}_4} f = 0$

and from the above calculation by replacing ε to $-\varepsilon$ we get

$$\int_{\tilde{\varphi}_2} f + \int_{\tilde{\varphi}_4} f = (-\varepsilon)e^{-\pi R^2}\int_0^1 e^{-\pi\varepsilon^2 t^2}\left[e^{-\pi 2\varepsilon tR} + e^{\pi 2\varepsilon tR}\right]dt$$

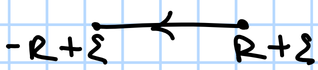
$$= -\int_{\varphi_2} f - \int_{\varphi_4} f$$

$$\Rightarrow \int_{\tilde{\varphi}_2} f + \int_{\tilde{\varphi}_4} f + \int_{\varphi_2} f + \int_{\varphi_4} f = 0 \quad \text{—— ③}$$

then $\displaystyle\left(\int_{\varphi_1} f + \int_{\varphi_2} f + \int_{\varphi_3} f + \int_{\varphi_4} f\right) + \left(\int_{\varphi_1} f + \int_{\tilde{\varphi}_2} f + \int_{\tilde{\varphi}_3} f + \int_{\tilde{\varphi}_4} f\right) = 0$

$$\Rightarrow 2\int_{\varphi_1} f + \int_{\varphi_3} f + \int_{\tilde{\varphi}_3} f = 0 \quad \text{—— ④} \quad (\because \text{ of ③})$$

now $\displaystyle\int_{\varphi_3} f = \int_R^{-R} e^{-\pi[t+\varepsilon]^2} dt$



$\varphi_3(t) = t+\varepsilon$

and $\tilde{\varphi}_3(t) = t - \varepsilon$

$-R-\varepsilon \longleftarrow R-\varepsilon$

$$\int_{\tilde{\varphi}_3} f = \int_{R}^{-R} e^{-\pi[t-\varepsilon]^2}\,dt$$

putting $t = -u$ (By substitution)

$$= \int_{-R}^{R} e^{-\pi[u+\varepsilon]^2}(-du)$$

$$= -\int_{R}^{-R} e^{-\pi[u+\varepsilon]^2}(-du)$$

$$= \int_{R}^{-R} e^{-\pi[u+\varepsilon]^2}\,du$$

putting $u = t$ (By substitution)

$$= \int_{R}^{-R} e^{-\pi[t+\varepsilon]^2}\,dt = \int_{\varphi_3} f$$

so $2\int_{\varphi_1} f + \int_{\varphi_3} f + \int_{\tilde{\varphi}_3} f = 0$

$$\Rightarrow 2\int_{\varphi_1} f + 2\int_{\varphi_3} f = 0$$

$$\Rightarrow \int_{\varphi_1} f + \int_{\varphi_3} f = 0$$

now $\varphi_1(t) = t$ $\quad$ $-R \longrightarrow R$

$$\int_{\varphi_1} f = \int_{-R}^{R} e^{-\pi t^2}\,dt$$

$$\int_{\varphi_3} f = \int_{R}^{-R} e^{-\pi[t+\varepsilon]^2}\,dt$$

$$= -\int_{-R}^{R} e^{-\pi[t+\varepsilon]^2}\,dt$$

adding both we get:

$$0 = \int_{-R}^{R} e^{-\pi t^2}\,dt - \int_{-R}^{R} e^{-\pi[t^2+\varepsilon^2+2\varepsilon t]}\,dt$$

as ε was any complex number replace ε by $i\varepsilon$ to get

$$\int_{-R}^{R} e^{-\pi t^2} dt - \int_{-R}^{R} e^{-\pi[t^2+(i\varepsilon)^2+2i\varepsilon t]} dt = 0$$

$$\int_{-R}^{R} e^{-\pi t^2} dt - \int_{-R}^{R} e^{-\pi t^2} e^{\pi \varepsilon^2} e^{-2\pi i \varepsilon t} dt = 0$$

$$\Rightarrow \int_{-R}^{R} e^{-\pi t^2} dt = e^{\pi\varepsilon^2} \int_{-R}^{R} e^{-\pi t^2} e^{-2\pi i \varepsilon t} dt$$

now by uniform continuity

$$\lim_{R\to\infty} \int_{-R}^{R} e^{-\pi t^2} dt = \lim_{R\to\infty} e^{\pi\varepsilon^2} \int_{-R}^{R} e^{-\pi t^2} e^{-2\pi i\varepsilon t} dt$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\pi t^2} dt = e^{\pi\varepsilon^2} \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-2\pi i \varepsilon t} dt$$

now given $\int_{-\infty}^{\infty} e^{-\pi t^2} dt = 1$

so $$1 = e^{\pi\varepsilon^2} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i \varepsilon x} dx$$

$$\left(\because \begin{array}{l} t = x \\ dt = dx \end{array} \right)$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i \varepsilon x} dx = e^{-\pi\varepsilon^2}$$

2. let $f(z) = \dfrac{z^2}{z^4+5z^2+6}$

then

$$\begin{aligned} z^4+5z^2+6 &= z^4+2z^2+3z^2+6 \\ &= z^2[z^2+2]+3[z^2+2] \\ &= [z^2+3][z^2+2] \end{aligned}$$

now $$f(z) = \frac{z^2}{(z^2+3)(z^2+2)}$$

$$= \frac{z^2[(z^2+3)-(z^2+2)]}{(z^2+3)(z^2+2)}$$

$$= \frac{z^2}{z^2+2} - \frac{z^2}{z^2+3}$$

$$= \frac{z^2+2-2}{z^2+2} - \frac{(z^2+3-3)}{z^2+3}$$

$$= 1 - \frac{2}{z^2+2} - \left(1 - \frac{3}{z^2+3}\right)$$

$$= \frac{3}{z^2+3} - \frac{2}{z^2+2}$$

so $f(z) = \dfrac{3}{z^2+3} - \dfrac{2}{z^2+2}$

$$= \frac{3}{(z+\sqrt{3}i)(z-\sqrt{3}i)} - \frac{2}{(z+\sqrt{2}i)(z-\sqrt{2}i)}$$

$$= \frac{3}{2\sqrt{3}i}\frac{[(z+\sqrt{3}i)-(z-\sqrt{3}i)]}{(z+\sqrt{3}i)(z-\sqrt{3}i)} - \frac{2}{2\sqrt{2}i}\frac{[(z+\sqrt{2}i)-(z-\sqrt{2}i)]}{(z+\sqrt{2}i)(z-\sqrt{2}i)}$$

$$= -\frac{i\sqrt{3}}{2}\left[\frac{1}{z-\sqrt{3}i} - \frac{1}{z+\sqrt{3}i}\right] + \frac{\sqrt{2}i}{2}\left[\frac{1}{z-\sqrt{2}i} - \frac{1}{z+\sqrt{2}i}\right]$$

so now let $f(z) = f_1(z) + f_2(z) + f_3(z) + f_4(z)$

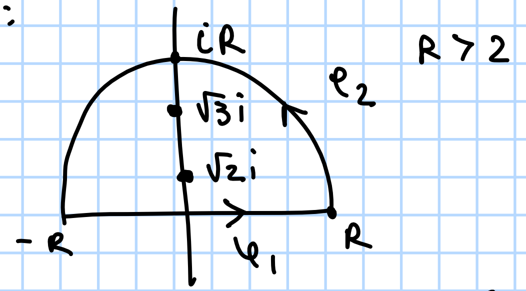
$$f_1(z) = -\frac{i\sqrt{3}}{2}\left(\frac{1}{z-\sqrt{3}i}\right)$$

$$f_2(z) = \frac{i\sqrt{3}}{2}\left(\frac{1}{z+\sqrt{3}i}\right)$$

$$f_3(z) = \frac{i\sqrt{2}}{2}\left(\frac{1}{z-\sqrt{2}i}\right)$$

$$f_4(z) = -\frac{i\sqrt{2}}{2}\left(\frac{1}{z+\sqrt{2}i}\right)$$

now let φ :



then $\displaystyle\int_\varphi f = \int_\varphi f_1 + \int_\varphi f_2 + \int_\varphi f_3 + \int_\varphi f_4$

now $\displaystyle\int_\varphi f_1 = \int_\varphi -\frac{i\sqrt{3}}{2}\left(\frac{1}{z-\sqrt{3}i}\right) = \left(\frac{-i\sqrt{3}}{2}\right)(2\pi i)\left[\frac{1}{2\pi i}\int_\varphi \frac{1}{z-\sqrt{3}i}\right]$

then we can apply residue theorm

to get $\displaystyle\frac{1}{2\pi i}\int_\varphi \frac{1}{z-\sqrt{3}i} = \mathrm{res}\left(\frac{1}{z-\sqrt{3}i}, \sqrt{3}i\right) = 1$

as $\tilde{f}_1(z) = \dfrac{1}{z-\sqrt{3}i}$, it is trivial to see it has pole at $z = \sqrt{3}i$

and also as $\tilde{f}_1(z) = \dfrac{1}{z-\sqrt{3}i}$

$\mathrm{Res}(\tilde{f}_1, \sqrt{3}i) = 1$

so $\displaystyle\int_\varphi f_1 = \pi\sqrt{3}$

Similary we can apply residue theorm for

$\tilde{f}_3(z) = \dfrac{1}{z-\sqrt{2}i}$ to get $\displaystyle\int_\varphi f_3 = -\pi\sqrt{2}$

now for f_2 and f_4 the integral is zero as

$\frac{1}{2\pi i}\int_{\varphi}\frac{1}{z+\sqrt{2}i}$ = winding no of $(-\sqrt{2}i)$ wrt φ, as $-\sqrt{2}i$ is outside $=0$

similarly for f_4

so $\int_{\varphi} f = (\sqrt{3}-\sqrt{2})\pi$ $\left(\because \int_{\varphi} f_1 = \pi\sqrt{3},\ \int_{\varphi} f_2 = 0,\ \int_{\varphi} f_3 = -\pi\sqrt{2},\ \int_{\varphi} f_4 = 0\right)$

now $\int_{\varphi} f = \int_{\varphi_1} f + \int_{\varphi_2} f$

where $\varphi_1(t) = t$

then

$$\int_{\varphi_1} f(\varphi_1(t))\,\varphi_1'(t)\,dt = \int_{-R}^{R} f(t)\,dt$$

and $\varphi_2(t) = Re^{it}$ $0 \le t \le \pi$

$\varphi_2'(t)\,dt = Re^{it}(i)\,dt$

$$\int_{\varphi_2} f = \int_0^{\pi} f(Re^{it})(Re^{it})(i)\,dt$$

$$= \int_0^{\pi} \frac{(Re^{it})^2\,(Re^{it})\,(i)\,dt}{(Re^{it})^4 + 5(Re^{it})^2 + 6}$$

$$= \frac{1}{R}\int_0^{\pi} \frac{(e^{it})^3(i)\,dt}{(e^{it})^4 + \frac{5}{R}(e^{it})^2 + \frac{6}{R^3}}\ dt$$

now $\left|(e^{it})^4 + \frac{5}{R}(e^{it})^2 + \frac{6}{R^3}\right| \geq 1 + \frac{5}{R} + \frac{6}{R^3} > 1$ —— ①

so $\left|\int_{\varphi_2} f\right| \leq \frac{1}{R}\int_0^{\pi}\left|\frac{(e^{it})^3(i)}{(e^{it})^4 + \frac{5}{R}(e^{it})^2 + \frac{6}{R^3}}\right| |dt|$

$\leq \frac{1}{R}\int_0^{\pi} |dt| = \frac{1}{R}\pi$ $\left(\because \text{from ① and } |e^{it}| = 1\right)$

so $\left|\int_{\varphi_2} f\right| \leq \frac{\pi}{R}$

now

$\left|\lim_{R\to\infty}\int_{\varphi_2} f\right| = 0 \Rightarrow \lim_{R\to\infty}\int_{\varphi} f = 0$ (from uniform continuity) —— ②

again from uniform continuity

$$\lim_{R\to\infty}\int_{\varphi_1} f + \lim_{R\to\infty}\int_{\varphi_2} f = \pi[\sqrt{3}-\sqrt{2}]$$

$\Rightarrow \lim_{R\to\infty}\int_{-R}^{R} f(t)\,dt + 0 = \pi[\sqrt{3}-\sqrt{2}]$ $(\because \text{from ② } = 0)$

$$\Rightarrow \int_{-\infty}^{\infty} f(t)\,dt = \pi[\sqrt{3}-\sqrt{2}] \quad (\because \text{uniform continuity})$$

let $t = x$

now $$\int_{-\infty}^{\infty} \frac{x^2}{x^4+5x^2+6}\,dx = \pi[\sqrt{3}-\sqrt{2}]$$

let $$I_2 = \int_{-\infty}^{0} \frac{x^2}{x^4+5x^2+6}\,dx \qquad I_1 = \int_{0}^{\infty} \frac{x^2}{x^4+5x^2+6}\,dx$$

then putting $u=-x$ (substitution) in I_2
we get $du = -dx$

$$I_2 = -\int_{\infty}^{0} \frac{u^2}{u^4+5u^2+6}\,du$$

$$= \int_{0}^{\infty} \frac{u^2}{u^4+5u^2+6}\,du$$

putting $u = x$ again

$$I_2 = \int_{0}^{\infty} \frac{x^2}{x^4+5x^2+6}\,dx = I_1$$

now $$\int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{\infty} f(t)\,dt$$

$$= I_2 + I_1$$

$$= I_1 + I_1 \quad (\because I_2 = I_1)$$

$$= 2I_1$$

so $$I_1 = \frac{1}{2}\int_{-\infty}^{\infty} f(t)\,dt$$

$$= \frac{1}{2}\left[\pi[\sqrt{3}-\sqrt{2}]\right]$$

so $$\int_{0}^{\infty} \frac{x^2}{x^4+5x^2+6}\,dx = \frac{\pi}{2}[\sqrt{3}-\sqrt{2}]$$

3. 

d is s.t φ is not winding it as φ does not wind d clockwise or anticlockwise so winding no of d wrt $\varphi = 0$

now winding no of c wrt φ is 1 as φ winds c once in the anticlockwise sense winding no of b wrt φ is 2 as φ winds b twice, both in anticlockwise sense.

winding no of a wrt φ is 0 as φ winds a twice, one in clockwise and one in anticlockwise this cancels out to get 0.

so $n(\varphi, a) = 0$

$n(\varphi, b) = 2$

$n(\varphi, c) = 1$

$n(\varphi, d) = 0$

Tutorial-12:

12.1 $\gamma \rightarrow$ Rectifiable curve
$\varphi \rightarrow$ function, cont on $\{\gamma\}$

$$F_m(z) = \int_\gamma \frac{\varphi(\omega)}{(\omega - z)^\gamma} d\omega \text{ for } z \notin \{\gamma\}$$

To prove: F_m is hol on $\mathbb{C} \setminus \{\gamma\}$ and $F_m'(z) = m F_{m+1}(z)$

proof: as $z \notin \{\gamma\}$
$\omega - z \neq 0 \ \forall \omega \in \{\gamma\}$
$\Rightarrow \frac{1}{(\omega - z)^m}$ is hol

now $\varphi(\omega)$ is cont for $\omega \in \{\gamma\}$

and so $\frac{\varphi(\omega)}{(\omega - z)^m}$ is cont $\forall \omega \in \{\gamma\}$

now $\lim_{h \to 0} \frac{F_m(z+h) - F_m(z)}{h}$

let $z+h$ be s.t as $h \to 0$
$z+h \notin \{\gamma\}$

then $\lim_{h \to 0} \frac{F_m(z+h) - F_m(z)}{h}$

$$= \lim_{h \to 0} \int_\gamma \frac{\varphi(\omega)}{h} \left[\frac{1}{(\omega - z - h)^m} - \frac{1}{(\omega - z)^m} \right] d\omega$$

$$= \lim_{h \to 0} \int_\gamma \frac{\varphi(\omega)}{h} \frac{1}{\underbrace{(\omega - z - h)^m}_{A^m} \underbrace{(\omega - z)^m}_{B^m}} \left[(\omega - z)^m - (\omega - z - h)^m \right] d\omega$$

$A^m - B^m$
$= (A - B)(A^{m-1}B^0 + A^{m-2}B^1 + \dots A^0 B^{m-1})$

$$= \lim_{h \to 0} \int_\gamma \frac{\varphi(\omega)}{h} h \times \frac{1}{A^m B^m} \times [A^{m-1}B^0 + A^{m-2}B^1 + \dots A^0 B^{m-1}] d\omega$$

$$= \int_\gamma \varphi(\omega) \lim_{h \to 0} \frac{1}{A^m} \frac{1}{B^m} [A^{m-1}B^0 + \dots A^0 B^{m-1}] d\omega$$

By uniform continuity

$$= \int_\gamma \varphi(\omega) \times \frac{m}{(\omega - z)^{m+1}} d\omega$$

$$= m \int_\gamma \frac{\varphi(\omega)}{(\omega - z)^{m+1}} d\omega$$

$= m F_{m+1}(z)$ ← continuous (trivial)

so $F_m(z)$ is hol $\forall z \in \mathbb{C} \setminus \{\gamma\}$
&
$F_m'(z) = m F_{m+1}(z)$

12.2 given $R>0$

<u>To prove</u>: $\forall \varepsilon>0, \exists z \in D(0,\varepsilon)$ s.t $|e^{1/z}|=R$

<u>proof</u>: let $z=x+iy$

then $e^{1/z}=e^{1/(x+iy)}$

$= e^{(x-iy)/(x^2+y^2)}$

$= e^{x/x^2+y^2}e^{-iy/x^2+y^2}$

$z = \dfrac{1}{\log \alpha + 2\pi i n}$

$\dfrac{1}{z} = \log \alpha + 2\pi i n$

$e^{\frac{1}{z}} = \alpha$

for n large enough

$n \to \infty$

$z \longrightarrow 0$

$|e^{1/z}| = |e^{x/x^2+y^2}|\,|1|$

$= e^{x/x^2+y^2}$

then for $\dfrac{x}{x^2+y^2} = \ln(R)$

$x = h^2 \ln(R)$

$y = h$

$\lim\limits_{h\to 0} \dfrac{x}{x^2+y^2} = \dfrac{h^2\ln(R)}{h^4\ln^2(R)+h^2} = \dfrac{\ln(R)}{h^2\ln^2(R)+1}$

$= \ln(R)$

so $z = (h^2\ln(R), h)$ for $h \to 0$

$h \in \mathbb{R}$

then $\lim\limits_{h\to 0} |e^{1/z}| = \lim\limits_{h\to 0} e^{\ln(R)} = e^{\ln(R)} = R$

now for $\delta>0$, $z \in D(0,\delta)$

$e^{1/z} = \alpha \quad \forall \alpha \in \mathbb{C}$

now, $e^{1/z} = (e^{x/x^2+y^2})(e^{-iy/x^2+y^2})$

$= |\alpha| e^{i\theta}$

$\theta \in [0, 2\pi)$

$\dfrac{x-iy}{x^2+y^2} = \ln|\alpha| + i\theta$

$\dfrac{-iy}{x^2+y^2} = i\theta$

$\dfrac{x}{x^2+y^2} = \ln|\alpha|$

$x = h^2\ln|\alpha|$

$y = h$

$\dfrac{-hi}{h^4\ln^2|\alpha|+h^2}$

$h = \dfrac{1}{\theta+2\pi n}$

$n\to\infty \quad h \longrightarrow 0$

$\dfrac{-i}{h^3\ln^2|\alpha|+h}$

now let $h = \frac{1}{-\theta + 2\pi n}$

then as $n \to \infty$
$h \to 0$

so,

$$\lim_{h \to 0} e^{\frac{x-iy}{x^2+y^2}} = |\alpha| \; e^{-iy/x^2+y^2}$$

$$= |\alpha| \; e^{i\theta - 2\pi n i}$$

as $n \to \infty$
but $n \in \mathbb{Z}$

$$e^{\frac{x-iy}{x^2+y^2}} = |\alpha| \; e^{i\theta}$$

$$e^{1/z} = \alpha$$

so, $\exists \, z \in D(0,\delta)$ s.t $e^{1/z} = \alpha$

12.3

(a) <u>test-1</u> : $\lim_{z \to 0} (\cancel{z}) \left(\frac{\log(z+1)}{z\cancel{z}} \right)$

$$= \lim_{z \to 0} \frac{\log(z+1)}{z}$$

now for $\log(z+1) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \ldots\ldots$

$$= \lim_{z \to 0} 1 - \frac{1}{2}z + \frac{1}{3}z^2 \ldots\ldots$$

$$= 1 \neq 0$$

so then $z=0$ is not removable

<u>test-2</u> : $\lim_{z \to 0} \left| \frac{\log(z+1)}{z^2} \right| \geq \lim_{z \to 0} \frac{|z - 1/2 z^2|}{|z^2|}$

$$\geq \lim_{z \to 0} \left| \frac{1}{z} - \frac{1}{2} \right|$$

so as $z \to 0$ $\left| \frac{\log(z+1)}{z^2} \right| \to \infty$

$\therefore$ at $z=0$ it is a pole

now to find the principle part:

$$f(z) = \frac{\log(z+1)}{z^2}$$

$$\tilde{f}(z) = \begin{cases} 0 \; ; \; z = 0 \\ \frac{z^2}{\log(z+1)} \end{cases} \longrightarrow \text{hol in a nbd of } 0$$

nbd of 0

now $\tilde{f}(z) = (z)^n \, h(z)$

$$= (z)^2 \left[\frac{1}{z - \frac{z^2}{2} + \frac{z^3}{3} \ldots} \right]$$

$$= z \left[\frac{1}{1 - \frac{z}{2} + \frac{z^2}{3} \ldots\ldots} \right]$$

$\longrightarrow h(z)$ is

so $f(z) = \frac{1}{z}\left[1 - \frac{z}{2} + \frac{z^2}{3} - \frac{z^3}{4} \cdots\cdots\right]$

$$f(z) = \frac{1}{z} - \frac{1}{2} + \frac{z}{3} - \frac{z^2}{4} + \frac{z^3}{5} \cdots$$

so, principal part $= \frac{1}{z}$

(b) $f(z) = z \sin\left(\frac{1}{z}\right)$

$$\lim_{z \to 0} (z)(z) \sin\frac{1}{z} = \lim_{z \to 0} z^2 \left[\frac{e^{i/z} - e^{-i/z}}{2i}\right]$$

now for $z = \frac{i}{2n\pi}$

$$= \lim_{z \to 0} z^2 \left[\frac{e^{2n\pi} - e^{-2n\pi}}{2i}\right]$$

as $n \to \infty$ $e^{2n\pi} \to \infty$

so $\lim \to \infty$

but for $z = \frac{1}{2n\pi} \to 0$ so not a removable

$\lim_{z \to 0} z \sin\frac{1}{z} \to 0$ for $z = \frac{1}{2\pi n}$ so not a pole

$\therefore$ it is essential

(c) $f(z) = \frac{\sin(z)}{z}$

$$\lim_{z \to 0} (\cancel{z-0}) \frac{\sin(z)}{\cancel{z}} = \lim_{z \to 0} \sin(z)$$

$= \sin(0)$ ($\because$ continuous)

$= 0$

so, removable at $z = 0$

now, $\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{\sin(z)}{z}$

$$= \lim_{z \to 0} \frac{e^{iz} - e^{-iz}}{2iz}$$

$$= \lim_{z \to 0} \frac{e^{2iz} - e^0}{e^{iz}\, 2iz}$$

$$= \lim_{z \to 0} \left(\frac{e^{2iz} - e^0}{2iz}\right) \frac{1}{e^{iz}}$$

$$= \frac{1}{e^0} \times \frac{(2i)\, e^0}{2i}$$

$$= \frac{2i}{2i} = 1$$

so $\lim_{z \to 0} \frac{\sin(z)}{z} = 1$

Tutorial-13 :

13.2 Suppose $f \in H(D(z_0,r)\setminus\{z_0\})$ and

$|f(z)| \leq A(z-z_0)^{-1+\varepsilon}$ for some $\varepsilon, A > 0$ $\forall$ $z \in D(z_0,r)\setminus\{z_0\}$

$\Rightarrow z_0$ is removable sing

$\lim_{z \to z_0} (z-z_0) f(z) = 0$ for z_0 removable

given $|f(z)| \leq A(z-z_0)^{-1+\varepsilon}$

$\Rightarrow |(z-z_0)||f(z)| \leq A|z-z_0|^{\varepsilon} \longrightarrow 0$ as $z \to z_0$

$\Rightarrow \lim_{z \to z_0} (z-z_0) f(z) = 0$

$\Rightarrow z_0$ is removable singularity

13.3 $\displaystyle\int_{\delta D(0,2)} \frac{z^2-2z-1}{(z^2+1)(z-1)}\, dz$

$\delta D(0,2)$ → clockwise

$$\frac{z^2-2z-1}{(z^2+1)(z-1)} = \frac{Az+B}{z^2+1} + \frac{C}{z-1}$$

$Az^2 - Az + Bz - B + Cz^2 + C$
$= z^2 - 2z - 1$

$A + C = 1$

$C - B = -1$

$-A + B = -2$

$$= \frac{2z}{z^2+1} - \frac{1}{z-1}$$

$$\int_{\delta D(0,2)} \frac{2z}{z^2+1} - \int_{\delta D(0,2)} \frac{1}{z-1}$$

→ $z = \pm i$ in $D(0,2)$

$z = 1$ in $D(0,2)$

So by residue theorm $-2\pi i(2-1) = -2\pi i$

13.4 __ID__ : $a, b \in$ ID

then $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$

for ring : $\exists$ additive & multiplicative identity

0 1

$f(z) = 0$ $\forall$ $z \in \mathbb{C}$ (AI)

$f(z) = 1$ $\forall$ $z \in \mathbb{C}$ (MI)

$f \in H(\Omega)$ $f \pm g \in H(\Omega)$
$g \in H(\Omega)$

$f \cdot g \in H(\Omega)$

$f \cdot 1 = f$

$f + 0 = f$ $\Rightarrow H(\Omega)$ is ring

__ID__ :

$f \in H(\Omega), g \in H(\Omega)$ s.t

$f \cdot g = 0$ then lets say $f \neq 0$, $g \neq 0$

$\exists z_0$ s.t $f(z_0) \neq 0$

as $f \in H(\Omega) \Rightarrow f \in C(\Omega)$

$\Rightarrow \exists U \ni z_0$ s.t $f(U) \neq 0$

then as $f(z)g(z)=0$
& $f(z)\neq 0 \ \forall z\in U$
$\Rightarrow g(z)=0 \ \forall z\in U$
$\Rightarrow g\equiv 0$ as $g(z)=0 \ \forall z\in U$

$\Rightarrow H(\Omega)$ is ID

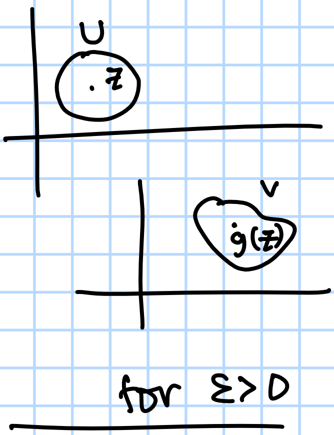
13.5 If Ω is not connected then $H(\Omega)$ is not an integral domain

$\Omega = D_1 \sqcup D_2 \quad D_1\cap D_2=\emptyset$
$f|_{D_1}\equiv 1 \quad g|_{D_1}\equiv 0$
$f|_{D_2}=0 \quad g|_{D_2}\equiv 1$
so $f\cdot g=0 \ \forall z\in\Omega$
but non-idd zero
$\Rightarrow H(\Omega)$ not ID

13.1 given $f\in H(\mathbb{C})$
$\hookrightarrow$ Holomorphic on $\mathbb{C}$
and also f is one-one

now if $f(z)=a_0+a_1z+a_2z^2+\cdots$
$\hookrightarrow$ expasion along $z_0=0$
as f is Hol this can be done
then
$g(z)=f\left(\frac{1}{z}\right)=a_0+\frac{a_1}{z}+\frac{a_2}{z^2}+\cdots$
if we assme that this has infinite terms then
$g(z)$ has an essential singularity at $z=0$

now as $g'(z)=f'\left(\frac{1}{z}\right)\left(\frac{-1}{z^2}\right) \ \forall z\in\mathbb{C}\setminus\{0\}$
now as f is not zero everywhere
$\exists z\in\mathbb{C}\setminus\{0\}$ s.t
$g'(z)\neq 0$
so, $|g'(z)|\neq 0$
and in a nbd of z not contaings 0
say U
s.t g is C^∞ on U
arg $g=u+iv$
so
$g:U\to V$ ← open
where $V=g(U)$
and $g(z)\in V$



then by inverse function theorm
g has inverse and
as onto it is bijective for
$g:U\to V$

now as $\overline{g(D(0,\varepsilon)\setminus\{0\})}=\mathbb{C}$ for $\varepsilon>0$ but this means
$g(D(0,\varepsilon)\setminus\{0\})\supseteq V$ $\divideontimes$

so $z=0$ not an essential singularity

so, $f(z)=a_0+a_1z+\dots a_nz^n$ (finite terms)

now $f(z)=a_0+a_1z+\dots a_nz^n$
$= c(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$
but as one-one
$f(\alpha_1)=f(\alpha_2)=0$
$\Rightarrow \alpha_1=\alpha_2$
$\Rightarrow f(z)=c(z-\alpha)^n$

$f(\alpha+1)=c$
$f(\alpha+\omega)=c(\omega)^n=c$
so $\omega^n=1$
$\Rightarrow n=1$
then $\omega=1$
so $f(z)=c(z-\alpha)$
$f(z)=a_0+a_1z$

13.2 $|f(z)|\leq A|z-z_0|^{-1+\varepsilon}$

now $|(z-z_0)f(z)|\leq A|z-z_0|^{\varepsilon}$
then $\lim\limits_{z\to z_0}|(z-z_0)f(z)|\leq \lim\limits_{z\to z_0} A|z-z_0|^{\varepsilon}\to 0$
$\Rightarrow \lim\limits_{z\to z_0}(z-z_0)f(z)=0$
$\Rightarrow z=z_0$ is removable singularity

13.3 $$\int_{\delta D(0,2)}\frac{z^2-2z-1}{(z^2+1)(z-1)}\,dz$$
$$=\int_{\delta D(0,2)}\frac{z^2-2z-1}{(z-i)(z+i)(z-1)}\,dz$$

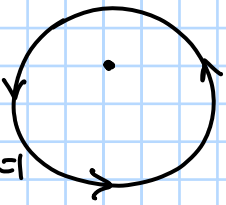
$$\frac{A}{(z-i)}+\frac{B}{(z+i)}+\frac{C}{(z-1)}=\frac{z^2-2z-1}{(z-i)(z+i)(z-1)}$$

$A(z+i)(z-1)+B(z-i)(z-1)+C(z^2+1)=z^2-2z-1$
$z=1$
$\Rightarrow C(2)=1-2-1=-2$
$\Rightarrow C=-1$
$z=i$
$A(2i)(i-1)+0+0=-1-2i-1$
$A(-2-2i)=-1-2i-1$
$A(-2-2i)=-2-2i$
$A=1$

$\cancel{z^2}-\cancel{z}+\cancel{iz}-\cancel{i}+B(\cancel{z^2}-\cancel{z}-\cancel{iz}+\cancel{i})-(\cancel{z^2}+1)=\cancel{z^2}-\cancel{2z}-\cancel{1}$

$B=1$

so $$\int_{\delta D(0,2)}\frac{1}{z-i}+\frac{1}{z+i}-\frac{1}{z-1}$$
$n(\delta D,i)=1$ $n(\delta D,1)=1$
$n(\delta D,-i)=1$



so $\frac{1}{2\pi i}\int_{\delta D(0,2)} \frac{1}{z-i} + \frac{1}{z+i} - \frac{1}{z-1}$

$= 1 + \cancel{1} - \cancel{1}$

$= 1$

$\Rightarrow \int_{\delta D(0,2)} \frac{1}{z-i} + \frac{1}{z+i} - \frac{1}{z-1} = 2\pi i$

as clockwise

$\Rightarrow \int_{-\delta D(0,2)} \frac{1}{z-i} + \frac{1}{z+i} - \frac{1}{z-1} = -2\pi i$

13.4 $\Omega \subseteq \mathbb{C}$

↳ connected & open

$H(\Omega)$ set of all holo funtions on Ω

<u>To prove</u>: $H(\Omega)$ is a ring and also ID

<u>proof</u>: ① $(H(\Omega),+)$ is an abelian group as

for $f, g \in H(\Omega)$

$f+g \in H(\Omega)$

(trivial)

$K \in H(\Omega)$

then $(f+g)+K = f+(g+K)$ (trivial)

$\exists f \equiv 0 \in H(\Omega)$ s.t

$g+0 = 0+g = g$

$f+g = g+f$ (trivial)

② $H(\Omega)$ is closed under multiplication

$f, g \in H(\Omega)$

then

$f \cdot g$ is also hol and $\in H(\Omega)$

as

$f = a_0 + a_1(z-z_0) + \cdots a_i(z-z_0)^i + \cdots$

$g = b_0 + b_1(z-z_0) + \cdots b_i(z-z_0)^i + \cdots$

$f \cdot g = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_i b_j (z-z_0)^{i+j} \in H(\Omega)$

③ $H(\Omega)$ is assosiative with multiplication

as

$f, g, h \in H(\Omega)$

$(f \cdot g) \cdot h = f \cdot (g \cdot h)$ (trivial)

④ distribuitive :

$f, g, h \in H(\Omega)$

$f \cdot (g+h) = f \cdot g + f \cdot h$ (trivial)

so $H(\Omega)$ is a ring

now if $f \cdot g \equiv 0 \ \forall z \in \Omega$

wlog $f \not\equiv 0$

then $\exists z_0 \in \Omega$ s.t

$f(z_0) \neq 0$

then $\exists U \ni z_0$

s.t $f(U) \neq 0$

but $f \cdot g \equiv 0 \ \forall z \in U$

$\Rightarrow g = 0 \ \forall z \in U$

$\Rightarrow g \equiv 0$ as Ω is connected & open

so if $f \cdot g \equiv 0$ then

$f \equiv 0$ or $g \equiv 0$

$\therefore H(\Omega)$ is ID

13.5 <u>To prove</u>: $\Omega \underset{\text{open}}{\subseteq} \mathbb{C}$, $H(\Omega)$ is ID $\Rightarrow \Omega$ is connected

<u>proof</u>:

lets say Ω is not connected, then $\exists D_1, D_2$ s.t

$$\Omega = D_1 \sqcup D_2$$
$$\phi = D_1 \cap D_2$$

let $f|_{D_1} \equiv 1 \quad f|_{D_2} \equiv 0 \in H(\Omega)$

$g|_{D_1} \equiv 0 \quad g|_{D_2} \equiv 1 \in H(\Omega)$ } trivial

but $f \cdot g = 0 \ \forall z \in D_1$ & $z \in D_2$

$\Rightarrow f \cdot g \equiv 0$

$\Rightarrow H(\Omega)$ is not ID

$\therefore \Omega$ not connected $\Rightarrow H(\Omega)$ not ID

$\Rightarrow H(\Omega)$ is ID $\Rightarrow \Omega$ is connected

Tutorial-14 :

14.1 $\sin(\pi z) = \pi z \prod_{n \geq 1} \left(1 - \frac{z^2}{n^2}\right)$

(a) $\cot(z) = \frac{\cos(z)}{\sin(z)}$

i.e $f(z) = \sin(z)$

$\cot(z) = \frac{f'(z)}{f(z)}$

$F(z) = \pi \cot(\pi z)$

To prove : $F(z+1) = F(z)$

proof: $F(z+1) = \pi \cot(\pi z + \pi)$

$= \pi \frac{\cos(\pi z + \pi)}{\sin(\pi z + \pi)}$

$= \pi \left[\frac{e^{i\pi z + i\pi} + e^{-i\pi z - i\pi}}{2}\right]\left[\frac{2i}{e^{i\pi z + i\pi} - e^{-i\pi z - i\pi}}\right]$

$= \pi \left[\frac{e^{i\pi z}[-1] + e^{-i\pi z}[-1]}{2}\right]\left[\frac{2i}{e^{i\pi z}[-1] - e^{-i\pi z}[-1]}\right]$

$= \pi \frac{\cos \pi z}{\sin \pi z}$

$= \pi \cot \pi z$

$= F(z)$

To prove : $F(z) = \frac{1}{z} + F_0(z)$ → Holomorphic near $z=0$

Proof : now lets find the pole of the function :

when $\sin \pi z = 0$

$\Rightarrow z = 0, \pm 1, \pm 2 \ldots$

so for $U \ni 0$ nbd of 0

we will have :

$F(z) = (z)^{-m} h(z)$ Holomorphic and non vanishing on U

now to calculate m:

$$\tilde{F}(z) = \begin{cases} 0 & ; z = 0 \\ \frac{1}{F(z)} & ; z \in U \setminus \{0\} \end{cases}$$

now $\tilde{F}(z)$ is hol and now

$\frac{1}{F(z)} = \frac{1}{\pi} \frac{\sin \pi z}{\cos \pi z}$

$= \frac{1}{\pi} \frac{[\pi z - (\pi z)^3/3! + \cdots]}{[1 - (\pi z)^2/2! + \cdots]} = z \frac{[\pi - \pi^3 z^2/3! + \cdots]}{\pi[1 - (\pi z)^2/2! + \cdots]}$

$= z\,h(z)$

$\underset{\text{Holo fuction}}{h(z)}$

$\Rightarrow \tilde{F}(z)$ has $z=0$ a zero of order 1

$\Rightarrow F(z)$ has pole of order 1

now so $F(z) = \frac{a_{-1}}{z} + \underbrace{a_0 + a_1 z + \cdots}_{\text{Holo function wrt } z=0}$

now $a_{-1} = \text{Res}(F(z), 0)$

now $\text{Res}(f(z), z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}\left[(z-z_0)^n f(z)\right]$

$$\Rightarrow \text{Res}(F(z), 0) = \lim_{z \to 0} \frac{1}{(0)!}\left(\frac{d}{dz}\right)^0 \left(z\,\pi\frac{\cos\pi z}{\sin\pi z}\right)$$

$$= \lim_{z\to 0} \frac{(\pi z)}{\sin\pi z}\cos(\pi z)$$

$$= \lim_{z\to 0} \frac{\pi z\left[1 - (\pi z)^2/2! + \cdots\right]}{\left[\pi z - (\pi z)^3/3! + \cdots\right]}$$

$\text{Res}(F(z), 0) = 1$

$\Rightarrow F(z) = \frac{1}{z} + F_0(z)$ ↙ Hol around $z=0$

$F_0(z) = a_0 + a_1 z + a_2 z^2 + \cdots$

<u>To prove</u>: $F(z)$ has poles of order 1 at integers and no other singularities

<u>proof</u>: $F(z) = \frac{\pi\cos\pi z}{\sin\pi z}$

so now $F(z)$ is not defined when $\sin(\pi z) = 0$

$\Rightarrow z = 0, \pm1, \pm2 \ldots$

or $z \in \mathbb{Z}$

$\therefore F(z)$ has singularities at $z \in \mathbb{Z}$

now to show they are poles of order 1

as $F(z+1) = F(z)$

$\forall z \in \mathbb{N} \Rightarrow F(z) = F(0)$

$\Rightarrow$ we can shift x axis s.t

Nbd of z_0 becomes nbd of 0

as 0 is a pole of order 1

so is z_0

$\Rightarrow$ true for all $z_0 \in \mathbb{Z}$

so all integers are poles of order 1

(b) $H(z) = \lim_{N\to\infty} \sum_{|n|\leq N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}$

<u>To prove</u> : $H(z) = H(z+1)$

<u>proof</u> : $H(z+1) = \frac{1}{z+1} + \sum_{n=1}^{\infty} \frac{2(z+1)}{(z+1)^2-n^2}$

$$= \lim_{N\to\infty} \sum_{|n|\leq N} \frac{1}{z+(1+n)}$$

$$= \lim_{N\to\infty} \frac{1}{z+(1-N)} + \frac{1}{z+(1-N+1)} + \cdots \frac{1}{z+(1+N)}$$

$$= \lim_{N\to\infty} \sum_{|1+n|\leq N} \frac{1}{z+(1+n)}$$

$$= \lim_{N\to\infty} \sum_{|n'|\leq N} \frac{1}{z+n'}$$

replace $1+n = n'$

$$= H(z)$$

<u>To prove</u> : $H(z) = \frac{1}{z} + H_0(z)$ ← Holomorphic near $z=0$

<u>proof</u> : As $H(z) = \lim_{N\to\infty} \frac{1}{z} + \sum_{1\leq |n|\leq N} \frac{1}{z+n}$

$$= \frac{1}{z} + \sum_{n\geq 1} \frac{2z}{z^2-n^2}$$

where as $2z$ is hol near $z=0$
and so is z^2-n^2
and on a small nbd $U \ni 0$
$z^2-n^2 \neq 0 \ \forall z \in U$
$\Rightarrow \frac{2z}{z^2-n^2}$ is hol

now for $|z|<1$ as $|z^2-n^2| \geq |n^2-z^2| > |n^2| + |z^2|$

$\Rightarrow |n^2-z^2| > |n^2| - |z^2|$

$|z^2| < 1$

$\Rightarrow -|z^2| > -1$

$\Rightarrow |n^2-z^2| > |n^2| - 1 > 2|n|^2$

for $n > 2$

$\Rightarrow \frac{1}{|n^2-z^2|} < \frac{1}{|n|^2-1} < \frac{1}{2|n|^2}$

$\Rightarrow \frac{2|z|}{|n^2-z^2|} < \frac{2}{|n|^2-1} < \frac{1}{2|n|^2}$

so for $\left\{ \frac{2z}{z^2-n^2} \right\}_{n\geq 2}$ $\exists\ C_n = \frac{1}{2|n|^2}$

s.t $|a_n| \leq C_n$ and $\sum C_n < \infty$

then by weierstrass – M test:

$\sum_{n \geq 2} \frac{2z}{z^2-n^2}$ is conv

$\Rightarrow \sum_{n \geq 1} \frac{2z}{z^2-n^2}$ is conv

$\Rightarrow \sum_{n \geq 1} \frac{2z}{z^2-n^2}$ is hol (By property of conv hol functions)

$\Rightarrow H_0(z)$ is hol around $z=0$

<u>To prove</u>: $H(z)$ has poles of order 1 at integers and no other singularity

<u>proof</u>: as $H(z) = \frac{1}{z} + H_0(z)$ around $z=0$

$\Rightarrow H(z) = \frac{1}{z} + a_0 + a_1 z + \cdots$

or $H(z)$ has pole at $z=0$ of order 1

now by shifting x axis s.t 0 becomes $z_0 \in \mathbb{Z}$

we can see $H(z)$, denominator term becomes 0 for $z \in \mathbb{Z}$

and then by shifting $\in \mathbb{Z}$

$H(z) = H(z+z_0)$

pole at $0 \Rightarrow$ pole at z_0 and order 1

(c) $\Delta(z) = F(z) - H(z)$

now $\Delta(z)$ is entire function as for $z \notin \mathbb{Z}$

$F(z)$, $H(z)$ are hol $\Rightarrow \Delta(z)$ is hol

and for $z \in \mathbb{Z}$

$F(z) = \frac{1}{(z-z_0)} + F_{z_0}(z)$ ← Hol near z_0

$H(z) = \frac{1}{(z-z_0)} + H_{z_0}(z)$ ← Hol near z_0

$\Rightarrow F(z) - H(z) = F_{z_0}(z) - H_{z_0}(z) = \Delta(z)$ ← Hol z near z_0

now, $\Delta(z+1) = F(z+1) - H(z+1)$
$= F(z) - H(z)$
$= \Delta(z)$

so, $\Delta(z)$ is hol $\forall z \in \mathbb{C}$ and $\Delta(z) = \Delta(z+1)$

$$\cot(\pi z) = i \frac{e^{-2\pi y} + e^{-2\pi i x}}{e^{-2\pi y} - e^{-2\pi i x}}$$

$$\lim_{z \to 0} \left| \pi \cot(\pi z) - \frac{1}{z} \right| = 0$$

$\forall \varepsilon > 0, \exists \delta > 0$ s.t

$$\left| \pi \cot \pi z - \frac{1}{z} \right| < \varepsilon \quad \forall |z| < \delta$$

now, for $|z| < \delta$

now, $|\Delta(z)| = |F(z) - H(z)|$

$$= \left| \pi \cot \pi z - \frac{1}{z} - \sum \frac{2z}{z^2 - n^2} \right|$$

$$\leq \left| \pi \cot \pi z - \frac{1}{z} \right| + \left| \sum \frac{2z}{z^2 - n^2} \right|$$

putting $\varepsilon = 1$

$\exists \delta > 0$ s.t

for $|z| < \delta \Rightarrow |\Delta(z)| < 1 + \left| \sum \frac{2z}{z^2 - n^2} \right| < \infty$

as $\left| \sum \frac{2z}{z^2 - n^2} \right|$ is bounded

now for $|z| \geq \delta$ and $|z| \leq 9/10$ (if $\delta < 9/10$, or else done)

$$|\pi \cot(\pi z)| = \pi \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right|$$

as $\delta \leq |z| \leq 9/10$

$\sin \pi z$ does not become 0

so, $\exists z_0$ s.t $\sin \pi z$ is minimum

say $|\sin \pi z| \geq \varepsilon$

then $\frac{1}{|\sin \pi z|} \leq \frac{1}{\varepsilon}$

$$\Rightarrow \pi \frac{|\cos \pi z|}{|\sin \pi z|} \leq \frac{\pi |\cos \pi z|}{|\sin \pi z|} \leq \frac{\pi}{\varepsilon}$$

$$\Rightarrow |F(z)| \leq \frac{\pi}{\varepsilon}$$

now $|H(z)| = \left| \frac{1}{z} + \sum \frac{2z}{z^2 - n^2} \right|$

$$\leq \left| \frac{1}{z} \right| + \left| \sum \frac{2z}{z^2 - n^2} \right|$$

$$\leq \frac{1}{\delta} + \underbrace{\left| \sum \frac{2z}{z^2 - n^2} \right|}_{\text{as } |z| < 1} < \infty$$

so for $|z| \leq 9/10$

$\Rightarrow \Delta(z)$ is bounded on $|z| \leq 9/8$

as $\Delta(z) = \Delta(z+1)$

$\Rightarrow \Delta(z)$ is bounded on $z = x+iy$ s.t

$-9/10 \leq y \leq 9/10 \quad x \in \mathbb{R}$

now for $z = x+iy$ s.t

$y > 9/10$ or $< 9/10$

we can show that $F(z)$ is bounded as $z \notin \mathbb{Z}$ and

$H(z)$ is also bounded because of same

$\Rightarrow \Delta(z)$ is bounded $\forall z \in \mathbb{C}$

$\Rightarrow \Delta(z)$ is Hol on $\mathbb{C}$ and bounded

$\Rightarrow \Delta(z) \equiv 0$ ($\because$ Liouville's theorm

$\Rightarrow F(z) = H(z)$

(d) $G(z) = \dfrac{\sin(\pi z)}{\pi}$

$$P(z) = z \prod_{n\geq 1}\left(1 - \frac{z^2}{n^2}\right)$$

To prove : $P(z)$ converges

proof : let $F_n(z) = 1 - \dfrac{z^2}{n^2}$

for $n \in \mathbb{N}$

then

$1 - F_n(z) = \dfrac{z^2}{n^2}$

$\Rightarrow |1 - F_n(z)| \leq \left|\dfrac{z^2}{n^2}\right|$

let $c_n(z) = \left|\dfrac{z^2}{n^2}\right|$

then $\displaystyle\sum_{n=1}^{\infty} c_n(z) = |z|^2\left(\sum \frac{1}{n^2}\right) < \infty$

as $|z|^2$ is fixed

$\Rightarrow \prod F_n(z)$ converges $\forall z \in \mathbb{C}$

$\Rightarrow z \prod F_n(z)$ converge $\forall z \in \mathbb{C}$

To prove : $\dfrac{P'(z)}{P(z)} = H(z)$ for z away from $\mathbb{Z}$

proof : let $\{F_n(z)\}_{n\geq 1}$ then

$$\prod_{n\geq 1} F_n(z) \longrightarrow F(z) \text{ (say)}$$

then $z \prod_{n \geq 1} F_n(z) \longrightarrow zF(z)$

now $\frac{F'(z)}{F(z)} = \sum \frac{F_n'(z)}{F_n(z)}$ $F_n(z)$ are non-vanishing as z away from $\mathbb{Z}$

then $\frac{zF'(z) + F(z)}{zF(z)}$

$$= \frac{F'(z)}{F(z)} + \frac{1}{z}$$

$$= \frac{P'(z)}{P(z)} = \frac{1}{z} + \sum_{n \geq 1} \frac{F_n'(z)}{F_n(z)} \qquad F_n(z) = 1 - \frac{z^2}{n^2}$$

$$= \frac{1}{z} + \sum_{n \geq 1} \frac{\left(-\frac{2z}{n^2}\right)}{1 - \frac{z^2}{n^2}}$$

$$= \frac{1}{z} + \sum_{n \geq 1} \frac{2z}{z^2 - n^2}$$

$$\frac{P'(z)}{P(z)} = H(z)$$

(e) Now, $G(z) = \frac{\sin(\pi z)}{\pi}$

$$\frac{G'(z)}{G(z)} = \frac{\pi/\pi \cos(\pi z)}{\frac{\sin(\pi z)}{\pi}}$$

$$= \pi \cot(\pi z)$$
$$= F(z)$$

then now $\left(\frac{P(z)}{G(z)}\right)' = \frac{P'(z)}{G(z)} - \frac{P(z)\,G'(z)}{(G(z))^2}$

$$= \frac{P'(z)}{P(z)} \times \frac{P(z)}{G(z)} - \frac{P(z)\,G'(z)}{(G(z))^2}$$

$$= H(z) \times \frac{P(z)}{G(z)} - \frac{P(z)}{G(z)} \times F(z)$$

$$= \frac{P(z)}{G(z)}(-\Delta(z))$$

$$\left(\frac{P(z)}{G(z)}\right)' = 0 \text{ for } z \notin \mathbb{Z}$$

$$\Rightarrow P(z) = cG(z) \text{ for } z \notin \mathbb{Z}$$

and for $z \in \mathbb{Z} \Rightarrow P(z) = 0$
& $G(z) = 0 \Rightarrow P(z) = cG(z) \;\forall z \in \mathbb{C}$

now $\lim_{z \to 0} \frac{P(z)}{G(z)} = c$

then $\lim_{z \to 0} \dfrac{z \prod_{n \geq 1} \left(1 - \frac{z^2}{n^2}\right)}{\frac{\sin \pi z}{\pi}}$

$$= \lim_{z \to 0} \frac{(\pi z)}{\sin(\pi z)} \prod_{n \geq 1} \left(1 - \frac{z^2}{n^2}\right)$$

$$= 1 = c$$

$$\Rightarrow P(z) = G(z) \quad \forall z \in \mathbb{C}$$

$$\Rightarrow \pi z \prod_{n \geq 1} \left(1 - \frac{z^2}{n^2}\right) = \sin(\pi z)$$

14.2 $|z| \leq \frac{1}{2}$

<u>To prove</u>: $|\log(1+z)| \leq 2|z|$

<u>proof</u>: $\log(1+z) = \int \frac{1}{1+z} dz$

where $\frac{1}{1+z} = 1 - z + z^2 - z^3 + z^4 \ldots$

$$\int \frac{1}{1+z} dz = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} \ldots$$

$$\Rightarrow \log(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$$

now, $|\log(1+z)| = \left| \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n} \right|$

$$\leq |z| \sum_{n=0}^{\infty} \frac{|z|^n}{n+1}$$

$$\leq |z| \sum_{n=0}^{\infty} \frac{1}{(2)^n (n+1)}$$

$$\leq |z| \sum_{n=0}^{\infty} \frac{1}{(2)^n} = 2|z|$$

$$\Rightarrow |\log(1+z)| \leq 2|z| \text{ for } |z| \leq \frac{1}{2}$$

14.3 $\max_{z \in D(0,1)} |f(z)|$

where $f(z) = z^2 + 2z - 1$

$$D(0,1) = \{ z \mid |z| \leq 1 \}$$

now by maximum modulus priniple, maxima not on int D as int D is open and

$f(z) = z^2 + 2z - 1$ is hol $\forall z \in \mathbb{C}$ (trivial)

$\Rightarrow |f(z)|$ has maxima on δD

now let $\gamma(t) = e^{2\pi i t}$ for $0 \leq t < 1$

then $\forall t \in [0,1)$ $\gamma(t) \in \delta D$ $\rightarrow$ paramatise δD

$$\max_{z \in D(0,1)} |f(z)| = \max_{t \in [0,1)} |f(\gamma(t))|$$

$$= \max_{t \in [0,1)} |\underbrace{e^{2\pi i t}}_{x_1 + iy_1} + \underbrace{2e^{i\pi t}}_{2x_2 + 2iy_2} - 1|$$

max when

$|x_1 + 2x_2 - 1| = |y_1 + 2y_2|$

this can be done by making

$|y_1 + 2y_2| = 2$

$y_2 = 1 \rightarrow x_2 = 0$

and $x_1 = -1 \rightarrow y_1 = 0$

so $e^{2\pi i t} = -1$

$e^{\pi i t} = i$

$\Rightarrow t = \frac{1}{2}$

and so for $t = \frac{1}{2}$

or for $z = e^{i\pi/2}$

$|f(z)|$ has max value of $2\sqrt{2}$

$z^2 + 2z - 1 = (z+1)^2 - 2$

$|f(e^{i\theta})| = |(e^{i\theta}+1)^2 - 1|$

$g(x,y) = x^2 + y^2 - 1$

$f(x,y) = |z^2 + 2z - 1|$
$= |(x+iy)^2 + 2(x+iy) - 1|$
$= |x^2 - y^2 + 2xyi + 2x + 2yi - 1|$
$= \left((x^2 - y^2 + 2x - 1)^2 + (2xy + 2y)^2\right)^{\frac{1}{2}}$

max $f(x,y)$ for $g(x,y) = 0$

$\nabla f = \lambda \nabla g$

$[(\cos(2\theta) + 2\cos\theta - 1)^2 + (\sin(2\theta) + 2\sin\theta)^2]^{\frac{1}{2}}$

$= 1 + 4 + 1 + 4\cos 2\theta \cos\theta - 2\cos(2\theta) - 4\cos\theta$
$+ 4\sin(2\theta)\sin\theta$

maximise $(6 - 2\cos 2\theta)^{1/2}$

$$\frac{df}{dx} = \frac{1}{2} \times \frac{1}{f(x,y)} \times [2(x^2 - y^2 + 2x - 1)(2x + 2) + 2(2xy + 2y)(2y)]$$

$$\frac{dg}{dx} = 2x$$

$|z^2 + 2z - 1| = \underbrace{|z|}_{\rightarrow 1} |z + 2 - \frac{1}{z}|$

$= |z - \bar{z} + 2|$
$= |2i\sin\theta + 2|$
$= 2|1 + i\sin\theta|$
$= 2(1 + \sin^2\theta)^{\frac{1}{2}}$ $\rightarrow$ maximum when $\sin\theta = 1$
$\Rightarrow \theta = \pi/2$

$\sin(z) = z - \frac{z^3}{3!} + \cdots$

then $f(z) = \frac{1}{\sin(z)}$

$= \frac{1}{z\left(1 - \frac{z^3}{3!} + \cdots\right)}$

let $= h(z)$ some hol function

$\sin(z) = z h(z)$

$z = 0 \Rightarrow h(z) = 1$

$\exists\, U \ni 0$ s.t $h(z) \neq 0 \;\forall\, z \in U$

$\Rightarrow f(z) = \frac{1}{z h(z)} = \frac{\tilde{h}(z)}{z}$

and $\lim_{z \to 0} z f(z) = \tilde{h}(0) = 1$

$\Rightarrow \operatorname{Res}(f, 0) = 1$